CERTAIN INTEGRAL AND FOURIER SERIES EQUATIONS AND THEIR APPLICATIONS

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Thesis

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By

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Under the Supervision of

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CERTIFICATE

This is certify that the matter embodied in this thesis entitled *"Certain Integral and Fourier Series Equations and Their Applications"* by **Devendra Singh** for the award of Ph.D. degree in Mathematics of the Bundelkhand University, Jhansi, is a record of research work carried out by him under my supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute of research for the award of any degree or diploma. Mr. Devendra Singh has put in more than 200 days attendance in the department of Mathematics of this college.

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PREFACE

This thesis entitled *"Certain Integral and Fourier Series Equations and Their Applications"* is an outcome of the researches carried out by me since 1997, under the kind supervision of **Dr. R.C. Singh Chandel**, Reader and Head of the Department of Mathematics, D.V. (P.G.) College, Orai. Now this work is being submitted for the award of Ph.D. degree in Mathematics.

The thesis comprises of seven chapters. First chapter is introductory giving various definitions and sources of integral and series equations. Second chapter deals with recent literature available on the topic. Third chapter finds solution of simultaneous dual integral equations. Triple integral equations are delt in chapter Four. Chapter Five and Six treat some series equations. Finally in chapter Seven we have given application of triple integral equations in a crack problem of elasticity. All references are included in the end.

Dated: 1st May, 2003

(Devendra Singh)

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I wish to express my deep sense of appreciation and gratitude to my thesis supervisor, ***Dr. R.C. Singh Chandel***, M.Sc., Ph.D., Reader and Head of the Department of Mathematics, for his constant encouragement, tireless guidance and inspiring discussions throughout the inception, execution and completion of this thesis. His constructive criticism and valuable suggestions have been a source of inspiration throughout this work.

My sincere thanks are due to ***Dr. N.D. Samadhiia***, Principal, D.V. (P.G.) College Orai and other authorities of the college for providing me necessary facilities for completing the thesis work.

I extend my sincere thanks to my wife ***Smt. Puspa Khochar*** with whose encouragement, co-operation and sacrifice, I have been able to complete my work.

Dated: 1st May, 2003

(Devendra Singh)

LIST OF RESEARCH PAPERS

The following is the list of research papers in which whole matter of thesis has been divided and is suitable for publication.

Papers Communicated:

1. Simultaneous dual integral equations associated with kernel of Fox.

2. Certain triple integral equations.

3. Some triple series equations with associated Legendre function.

4. Five series equations involving Jacobi polynomials.

5. Two Griffith cracks at the interface opened by forces at crack surfaces.

DEDICATION

This work is dedicated to my loving father, **Shri UMER SINGH** who with their unbounded energy and zest for hand work, always infused in me the spirit of courage, confidence and joy;

AND

to my mother, **Mrs. JAYAWATI DEVI**, an apostle of love and care, who took all pains to help me in pursuing my studies. God has truly blessed me with her parentage and companionship.

Dated: 1st May, 2003

(Devendra Singh)

CONTENTS

CHAPTER –1

INTRODUCTION

During the last seven decades the special type of integral equations have been proved to be very useful tool in the analysis of mixed boundary value problems in Mathematical Physics. The same is also true for series equations such as dual series equations, triple series equations, etc.

Integral equations occur in many fields of mechanics and mathematical physics. They also represent the formula for the solutions of differential equations. A differential equation can be replaced by an integral equation with the help of its boundary conditions. The solutions of these integral equations satisfy these boundary conditions.

1.1 SOURCE OF DUAL INTEGRAL EQUATIONS

One of the classic problems of mathematical physics is that of determining the potential in the field due to a circular disc at unit potential placed with its plane parallel to and equidistant from two earthed infinite parallel plates. The integral form of the solution found is well suited to numerical computation when the plates are fairly far apart. If we use cylindrical polar co-ordinates (r,z) we many take the disc to be specified by $z=0$, $r<1$ and the plates as $z = \pm h$. The potential has to satisfy Laplace's equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1.1.1}$$

with, since there is symmetry about the plane z=0,

$$\left.\begin{aligned} V = 1, \qquad & 0<r<1 \\ \frac{\partial V}{\partial z}=0, \qquad & r>1 \end{aligned}\right\} \text{ when } z=0 \tag{1.1.2}$$

and

$$V = 0, \quad 0 < r < \infty \quad \text{when } z = h \tag{1.1.3}$$

The Hankel transform of V denoted by $\overline{V}$ has to satisfy the ordinary differential equation

$$\frac{d^2\overline{V}}{\partial z^2}=\rho^2\overline{V} \tag{1.1.4}$$

The general solution of this equation is

$$\overline{V}=Ae^{\rho z}+Be^{-\rho z} \tag{1.1.5}$$

and, since from (1.1.3), $\overline{V} = 0$ when $z = h$, we get $A = -B\ e^{-\rho h}$.

Hence

$$\overline{V}=Be^{-\rho z}-e^{-\rho(z-2h)} \tag{1.1.6}$$

The inversion formula for Hankel transform is defined as

$$\bar{f}(r)=\int_0^\infty f(\rho)\rho\, J_n(r\rho)\,d\rho \tag{1.1.7}$$

This inversion formula then gives

$$V=\int_0^\infty \rho\, B\left[e^{-\rho z}-e^{-\rho(z-2h)}\right]J_0(r\rho)\,d\rho \tag{1.1.8}$$

and substitution in the boundary condition on z = 0, equation (1.1.2) yields

$$\left.\begin{aligned}&\int_0^\infty \rho B\left[1-e^{-2h\rho}\right]J_0(r\rho)\,d\rho=1, \qquad 0<r<1\\ &\int_0^\infty \rho^2 B\left[1+e^{-2h\rho}\right]J_0(r\rho)\,d\rho=0, \qquad r<1\end{aligned}\right\} \tag{1.1.9}$$

This is a set of dual integral equations, where B is the unknown function to be determined.

Thus there are several problems in potential theory which can be reduced to dual integral equations.

1.2 SOURCE OF DUAL SERIES EQUATIONS

If we make use of series solution of Laplace's equation we are led to a paid of dual relations involving series. For instance, suppose we wish to find the axisymmetric solution $V(\rho,z)$ of Laplace's equation in the semi-infinite cylinder $0 \le \rho \le a$, $z \ge 0$ satisfying the boundary conditions,

$$V(\rho,z)\to 0 \quad \text{as} \quad z\to\infty$$

$$V(a,z) = 0, \quad z \geq 0 \tag{1.2.1}$$

$$V(\rho,0) = f(\rho) \quad 0 \leq \rho \leq 1 \tag{1.2.2}$$

$$\left|\frac{\partial V}{\partial z}\right|_{z=0} = 0, \quad 1 < \rho \leq a \tag{1.2.3}$$

Then the harmonic function

$$V(\rho,z) = \sum_{=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_m \rho) e^{-\lambda} n^z \tag{1.2.4}$$

will satisfy the condition (1.2.1) provided that λ_I, i = 1 to ∞ are the positive zeros of $J_0(\lambda a)$. The conditions (1.2.2) and (1.2.3) are then equivalent to the paid of relations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n \rho) = f(\rho), \quad 0 \leq \rho < 1 \tag{1.2.5}$$

$$\sum_{n=1}^{\infty} a_n J_0(\lambda_n \rho) = 0, \quad 1 < \rho \leq a \tag{1.2.5}$$

where $[\lambda_n]$ is the sequence of positive zeros of $J_0(\lambda a)$. A pair of equations of this type is called a pair of dual series equations. Thus there are several problems in potential theory [192], which can be reduced to dual series equations.

1.3 DEFINITIONS OF INTEGRAL EQUATIONS

An integral equation is an equation in which an unknown function tobe

determined appears under one or more integral signs. If the derivatives of function are involved in it, then such type of equation is called an integro-differential equation.

An integral equation is called linear if only linear operations are performed upon the unknown function, e.g.

$$\phi(u)=\int_a^b K(u,y)g(y)dy \tag{1.3.1}$$

is a linear integral equation and

$$\phi(u)=\int_a^b K(u,y)\{g(y)\}^2 dy \tag{1.3.2}$$

is a non-linear integral equation.

As equation of the form

$$\alpha(u)\phi(u)=F(u)+\lambda\int_\Omega K(u,y)\phi(y)dy=0 \tag{1.3.3}$$

is called the linear integral equation where $\phi(u)$ is unknown function, $K(u,y)$ is the kernel of the integral equation, $F(u)$ and $\alpha(u)$ all are known functions, λ is non-zero real or complex number and Ω is the domain of the auxiliary variable y over which the integration extends.

Linear integral equations involve the integral operator

$$L = \int_{\Omega} K(u,y)dy$$

having the kernel K(u,y). For any constants c_1 and c_2 it satisfies the linearity condition

$$L[c_1\phi_1(y) + c_2\phi_2(y)] = c_1 L[\phi_1(y)] + c_2 L[\phi_2(y)] \tag{1.3.5}$$

where

$$L[\phi(y)] = \int_{\Omega} K(u,y)\phi(y)\,dy \tag{1.3.6}$$

1.4 CLASSIFICATION OF INTEGRAL EQUATIONS

Linear integral equations are classified into following types:

1.4.1 Fredholm Integral Equation

An equation having the domain of integration Ω fixed is called a Fredholm integral equation, e.g.

$$\alpha(u)\phi(u) = F(u) + \lambda \int_a^b K(u,y)\phi(y)\,dy \tag{1.4.1}$$

Fredholm integral equations are of two types:

(i) When $\alpha = 0$, then equation takes the form

$$F(u) = \lambda \int_a^b K(u,y)\phi(y)\,dy, \quad a \le u \le b \tag{1.4.2}$$

This type of equation, which involves the unknown function ϕ only under the integral sign is called Fredholm integral equation of first kind.

(ii) When $\alpha = 1$, then the equation takes the form

$$\phi(u) = F(u) + \lambda \int_a^b K(u, y)\phi(y)\,dy, \quad a \le u \le b \tag{1.4.3}$$

Such type of equations, which involve the unknown function ϕ both inside as well as outside the integral sign is called Fredholm integral equation of second kind.

(iii) When $\alpha = 1$ and $F(u) = 0$, then the equation reduces to

$$\phi(u) = \lambda \int_a^b K(u, y)\phi(y)\,dy, \quad a \le u \le b \tag{1.4.4}$$

This type of equation is known as the homogeneous Fredholm integral equation of second kind.

1.4.2 Volterra Integral Equation

An equation having variable upper limit of integration is called Volterra integral equation, e.g.

$$\alpha(u)\phi(u) = F(u) + \lambda \int_a^u K(u, y)\phi(y)\,dy \tag{1.4.5}$$

(i) When $\alpha = 0$, then the equation takes the form

$$F(u) = \lambda \int_a^u K(u, y) \phi(y) dy, \quad a > -\infty \tag{1.4.6}$$

This type of equation, which involves the unknown function ϕ only under the integral sign is called volterra's integral equation of first kind.

(ii) When $\alpha = 1$, then the equation takes the form

$$\phi(u) = F(u) + \lambda \int_a^u K(u, y) \phi(y) dy \tag{1.4.7}$$

Such type of equations which involve the unknown function ϕ inside as well as outside the integral sign is called the Volterra's integral equation of second kind.

(iii) When $\alpha =$ 1 and $f(u) = 0$, then the equation reduces to

$$\phi(u) = \lambda \int_a^u K(u, y) \phi(y) dy \tag{1.4.8}$$

This type of equation is known as the homogeneous Volterra's integral equation of second kind.

1.4.3 Singular Integral Equation

An integral equation either having the range of integration infinite or kernel has singularities within the range of integration is called singular integral equation. Such equations occur frequently in mathematical physice and process very unusual properties. One of the simplest integral equation is

the Able integral equation:

$$f(y)=\int_0^y \frac{g(x)}{(y-x)^\alpha}dx, \qquad 0<x<1 \tag{1.4.9}$$

The solution of this integral equation is given by

$$g(x)=\frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\left[\int_0^x f(y)(x-y)^{\alpha-1}dy\right] \tag{1.4.10}$$

The integral equation (1.4.9) is a special case of the singular integral equation

$$f(y)=\int_a^y \frac{g(x)dx}{[h(y)-h(x)]^\alpha}, \qquad 0<\alpha<1 \tag{1.4.11}$$

where h(x) is a strictly monotonically increasing and differentiable function in (a,b) and $h'(y)\neq 0$ in this interval. It solution is given by

$$g(x)=\frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\left[\int_0^x \frac{h'(t)f(t)dt}{\{h(x)-h(t)\}^{1-\alpha}}\right] \tag{1.4.12}$$

similarly, the integral equation

$$f(y)=\int_y^b \frac{g(x)dx}{[h(x)-h(y)]^\alpha}, \qquad 0<\alpha<1 \tag{1.4.13}$$

is such in which h(x) is monotonically increasing and $a<y<b$. This equatio has the solution

$$g(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx}\left[\int_x^b \frac{h'(t)f(t)dt}{\{h(x)-h(t)\}^{1-\alpha}}\right] \quad (1.4.14)$$

1.4.4 NON-LINEAR INTEGRAL EQUATION

An equation in which the unknown function appears under an integral sigh to a power n(n > 1), is known as a non-linear integral equation, e.g.

$$\phi(u) = F(u) + \lambda \int_a^b K(u,y)\phi(y)\,dy, \quad (1.4.15)$$

$$\phi(u) = F(u) + \lambda \int_a^b K\{u, y\, \phi(y)\}dy, \quad (1.4.16)$$

1.5 SPECIAL SETS OF INTEGRAL EQUATIONS

1.5.1 Dual Integral Equations

The pair of equations

$$\int_0^\infty G(y)\psi(y)k(x,y)dy = f(x), \quad 0 < x < 1 \quad (1.5.1)$$

$$\int_0^\infty \psi(y)k(x,y)dy = G(x), \qquad 1 < x < \infty \quad (1.5.2)$$

where G(y) is a function of y alone, is known as weight function ψ(y) is athe unknown function to be determined, K(x,y) is the known function known as kernel and f(x), g(x) are given functions in the given intervals, are known as "dual integral equations". In practical applications the kernels are usually

either of Bessel type $J_\nu(xy)$ or of trigonometric type, sing(xy), cos(xy).

These equations arise frequently in the solution of boundary value problems in which the conditions on one boundary is a mixed one.

1.5.2 Triple Integral Equations

Triple integral equations are of two kinds.

(i) Equations of the First kind:

The set of equations

$$\int_0^\infty \psi(y)k(x,y)dy = f(x), \qquad 0 < x < a \qquad (1.5.3)$$

$$\int_0^\infty G(y)\psi(y)k(x,y)dy = g(x), \qquad a < x < b \qquad (1.5.4)$$

$$\int_0^\infty \psi(y)k(x,y)dy = h(x), \qquad b < x < \infty \qquad (1.5.5)$$

where f(x), g(x) and h(x) are prescribed functions in the given interval, are known as triple integral equations of the first kind.

These equations arise in the solution of mixed boundary value problems of methematical physics in which different conditions are imposed on three parts of one of the boundries.

(ii) Equation of the Second kind:

$$\int_0^\infty G(y)\psi(y)k(x,y)dy = f(x), \qquad 0 < x < a \qquad (1.5.6)$$

$$\int_0^\infty \psi(y)k(x,y)dy = g(x), \qquad a < x < b \qquad (1.5.7)$$

$$\int_0^\infty G(y)\psi(y)k(x,y)dy = h(x), \qquad b < x < \infty \qquad (1.5.8)$$

where f(x), g(x) and h(x) are prescribed functions, are known as "triple integral equations" of the second kind.

Such types of equations arise in the solution of mixed boundary value problems of mathematical physics in which different conditions are imposed on three parts of one of the boundaries.

1.5.3 Higher Order Integral Equations

In the same way, higher order integral equations may be defined. In general, we now define "n-tuple integral equations".

(i) n-Tuple Integral Equations of the First kind

The set of "n-tuple integral equations" of first kind are as follows:

$$\int_0^\infty \psi(y)k(x,y)dy = f_i(x), \qquad a_{i-1} < x < a_i \qquad (1.5.9)$$

$$i = 1, 3, 5, \ldots\ldots\ldots \text{ and } a_0 = 0$$

$$\int_0^\infty G(y)\psi(y)k(x,y)dy = f_i(x), \qquad a_{i-1} < x < a_i \qquad (1.5.10)$$

$$i = 2, 4, 6, ...n \text{ and } a_n = \infty$$

where $f_i(x)$, i = 1,2,3, ... , n are known functions. Quadruple, 5-tuple and 6-tuple integral equations are special cases of these equations.

(ii) n-Tuple Integral Equations of the Second kind

The set of "n-tuple integral equations" of second kind are as follows:

$$\int_0^\infty G(y)\psi(y)k(x,y)dy = g_i(x), \qquad a_{i-1} < x < a_i \qquad (1.5.11)$$

$$i = 1, 3, 5, ...n - 1 \text{ and } a_0 = 0$$

$$\int_0^\infty \psi(y)k(x,y)dy = g_i(x), \qquad a_{i-1} < x < a_i \qquad (1.5.12)$$

$$i = 2, 4, 6, ...n \text{ and } a_n = \infty$$

where $g_i(x)$, i = 1,2,3, ... , n are known functions. Quadruple, 5-tuple and 6-tuple integral equations are special cases of these equations.

1.6 SPECIAL SETS OF SERIES EQUATIONS

1.6.1 Dual Series Equations

The pair of equations

$$\sum_{n=0}^{\infty} A_n G_n P(n,y) = f(y), \qquad 0 < y < a \qquad (1.6.1)$$

$$\sum_{n=0}^{\infty} A_n P(n, y) = g(y), \qquad a < y < \infty \tag{1.6.2}$$

where G_n is a known function of n and some other parameters, P(n,y) is the given polynomial of order n and argument y, and sequence (A_n) is to be found, are called "dual series equations" involving polynomial P(n,x).

These equations have been proved to be very useful tool for finding solutions to various mixed boundary value problems of elasticity.

1.6.2 Triple Series Equations

These equations are of two kinds:

(i) Triple Series Equations of the First kind

The following set of equations:

$$\sum_{n=0}^{\infty} A_n P(n, y) = f(y), \qquad 0 < y < a \tag{1.6.3}$$

$$\sum_{n=0}^{\infty} A_n G_n P(n, y) = g(y), \qquad a < y < b \tag{1.6.4}$$

$$\sum_{n=0}^{\infty} A_n P(n, y) = h(y), \qquad b < y < \infty \tag{1.6.5}$$

are known as "triple series equations of first kind". In such types of series equations f(y), g(y) and h(y) are known functions and the other symbols have their usual meaning.

(ii) Triple Series Equations of the Second kind

The following set of equations:

$$\sum_{n=0}^{\infty} A_n G_n P(n,y) = f(y), \qquad 0 < y < a \tag{1.6.6}$$

$$\sum_{n=0}^{\infty} A_n P(n,y) = g(y), \qquad a < y < b \tag{1.6.7}$$

$$\sum_{n=0}^{\infty} A_n G_n P(n,y) = h(y), \qquad b < y < \infty \tag{1.6.8}$$

are known as "triple series equations of second kind". Here, symbols have their usual meanings.

Both kinds of triple series equations have been used for finding the solution of problems of potential theory and crack problems of electricity.

1.6.3 Higher Order Series Equations

Similarly we have define quadruple, 5-ruple and 6-tuple series equations. In general, we now define "n-tuple series equations".

(i) n-Tuple Series Equations of First kind

The set of "n-tuple integral equations" of first kind are as follows:

$$\sum_{n=0}^{\infty} A_n P(n,y) = f_i(y), \qquad a_{i-1} < y < a_i \tag{1.6.9}$$

$$\text{where } i = 1, 3, \ldots\ldots\ldots n-1 \text{ and } a_0 = 0$$

$$\sum_{n=0}^{\infty} A_n G_n P(n,y) = f_i(y), \qquad a_{i-1} < y < a_i \qquad (1.6.10)$$

$$\text{where } i = 2, 4, \ldots\ldots\ldots n \text{ and } a_n = \infty$$

where the symbols have their usual meanings.

(ii) n-Tuple Series Equations of Second kind

The set of "n-tuple integral equations" of first second are as follows:

$$\sum_{n=0}^{\infty} A_n G_n P(n,y) = g_i(y), \qquad a_{i-1} < y < a_i \qquad (1.6.11)$$

$$\text{where } i = 1, 3, \ldots\ldots\ldots n-1 \text{ and } a_0 = 0$$

$$\sum_{n=0}^{\infty} A_n P(n,y) = g_i(y), \qquad a_{i-1} < y < a_i \qquad (1.6.12)$$

$$\text{where } i = 2, 4, \ldots\ldots\ldots n \text{ and } a_n = \infty$$

Quadryple, 5-tuple and 6-tuple series equations are some special cases of these equations.

1.7 APPLICATION OF INTEGRAL AND SERIES EQUATIONS

Integral and series equations have been extensively used for solving mixed boundary value problems of electrostatics, electricity, diffraction theory and fluid mechanics. References for such applications can be found in text book of Sneddon [192], Sneddon and Lowengrub [194] and Muskhelishvili [135]. For recent work on Applications some research paper cited in the Bibliography of this Thesis may be seen.

In the present Thesis we have used triple integral equations for finding solution of a crick problem in Chapter Seven.

CHAPTER –2

LITERATURE SURVEY

In this chapter, we give the historical development of the work done in solving different types of integral equations and Fourier series equations.

2.1 INTEGRAL EQUATIONS

2.1.1 Dual Integral Equations

(i) Dual Integral Equations Involving Bessel Function

Earlier research workers actually took interest on dual integral equations after the publication of the book 'An Introduction to the Fourier Integrals' by Titchmarsh [118]. In this text, the following dual integral equations involving Bessel functions were considered:

$$\int_0^\infty u^\alpha \{1+H(u)\}\psi(u)J_\nu(xu)du = f(x), \qquad 0<x<1 \tag{2.1.1}$$

$$\int_0^\infty \psi(u)J_\nu(xu)du = g(x), \qquad 1<x<\infty \tag{2.1.2}$$

with H(u) = g(x) = 0 and α = -1.

The range of the solution of the above equations were extended by Miss Busbridge [7]. The integral equation method for solving the above equations was given by Tranter [120]. He assumed the suitable integral

representation for the unknown function in terms of another unknown function and obtained the solution in the form of Fredholm integral equation, which can be solved numerically. Gordon [42] obtained the solution of the above equations by using certain Sonine's discontinuous integrals.

The solution of more general equations was given by Peters [85]. He took the equations:

$$\int_0^\infty t^\alpha \phi(t) J_\mu(xt)\,dt = f(x), \qquad 0 < x < 1 \tag{2.1.3}$$

$$\int_k^\infty \left(t^2 - k^2\right)^\beta \phi(t) J_\nu(xt)\,dt = g(x), \qquad 1 < x < \infty, k > 0 \tag{2.1.4}$$

Noble [72] solved the above equations by applying multiplying factor method. Dwivedi [18] obtained the solution of some other equations by applying the same method.

Fan [39] considered some dual integral equations and simultaneous equations and solved them by transferring the equations to general system of functional equations in the complex domain. He also gave the applications of these equations to solid mechanics and fluid mechanics.

Kalaba and Zagustin [47] solved some dual integral equations by initial value problem approach and obtained the solution in the form of a Fredholm integral equation. Tanno [117] also considered certain dual integral equations as convolution transforms.

Srivastav and Parihar [115] applied generalized function concept to obtain the solution of certain dual integral equations. Srivastav [116] considered dual integral equations with trigonometric kernels and tempered distribution. He also introduced L-2 theory to solve trigonometric equations.

Tranter [119] gave the solution of dual integral equations with Bessel function of zeroth order. Sneddon [102] considered dual integral equations with Bessel function of zeroth order and reduced them to an Abel-Schlömilch type of integral equation whose solution was well known.

Nguyen Van Ngok and Popov [70] considered the dual integral equations connected with Fourier transforms and solved them by applying the method of successive approximations.

Labedev and Uflyand [49] obtained the solution of dual equations of zeroth order in the form of Fredholm integral equation with symmetric kernel.

Recently Chakrabarti [8] considered the three different sets of dual integral equations involving Bessel functions of first kind and of order one.

Aizikovich [4] gave a method to reduce the dual integral equations into an infinite algebraic system. The kernels of these equations are the eigen functions of a Sturm-Liouville problem for a second order equation in terms of a parameter, which is naturally small.

Cherskii [9] has considered a multidimensional dual equations of convolution type

$$u(x) = \int_{\Omega+U\Omega_+} k_1(x-s)u(s)ds = g(x), \qquad (x \in \Omega_+) \qquad (2.1.5)$$

$$u(x) = \int_{\Omega+U\Omega_-} k_2(x-s)u(s)ds = g(x), \qquad (x \in \Omega_-) \qquad (2.1.6)$$

where k is defined on the space $L(M_m^n)$ of complex valued functions defined on M_m^n with finite norms and $M_m^n (0 \le m \le n)$ is the set of points (x_1, x_2, ..., x_m, x_{m+1}, ..., x_n) whose first m co-ordinates are real, while the others are integers. Using the factorization method the system is solved by quadratures. Moreover, k is considered as an absolutely integrable function, vanishing in the exterior of a set W.

Mandal [58] applied an elementary procedure based on Sonine's integral to reduce dual integral equations with Bessel functions of different orders as kernels and an arbitrary weight function, into a Fredholm integral equation of the second kind. The result obtained here is more general in nature and includes many results, concerning dual integral equations with Bessel functions as kernels, known in the literature.

Considering the three dimensional problems of the theory of elasticity with mixed boundary condition with circular lines of the boundary conditions, dual integral equations arise. Such types of dual integral equations considered by Abramyan [1] are

$$\int_0^\infty \beta^\alpha \psi(\beta) J_m(\beta r) d\beta = f(r), \qquad 0 < r < a \qquad (2.1.7)$$

$$\int_{k}^{\infty} \psi(\beta) J_m(\beta r) d\beta = g(r), \qquad r > a \qquad (2.1.8)$$

where $\alpha = \pm 1$, f(r) and g(r) are known functions in the prescribed intervals. $\psi(\beta)$ is the function to be determined and m = 0, 1, ... He solved these equations by the method of orthogonalisation of equations in the shorter way.

Rahman [88] found an effective polynomial solution to a class of dual integral equations which arise in many mixed boundary value problems in the theory of elasticity. The dual integral equations are first transformed into a Fredholm integral equation of the second king via an auxiliary function, which is next reduced to an infinite system of linear algebraic equations, by representing the unknown auxiliary function in the form of an infinite series of Jacobi polynomials. The approximate solution of this infinite system of equations can be obtained by a suitable truncation. It is shown that the unknown function involving the dual integral equations can also be expressed in the form of an infinite series of Jacobi polynomials with the same expansion coefficient with no numerical integration involved. The main advantage of the present approach is that the solution of the dual integral equations thus obtained in numerically more stable than that obtained by reducing them directly to an infinite system of equations, in so far as the expansion coefficients are determined essentially by solving a second kind integral equation.

(ii) Dual Integral Equations with Trigonometric Kernels

The following is the general form of dual integral equations with trigonometric kernels:

$$\int_0^\infty u^\alpha \psi(u) \begin{matrix}\sin\\ \cos\end{matrix}(xu)\,du = f(x), \qquad 0 < x < 1 \qquad (2.1.9)$$

$$\int_0^\infty \psi(u) \begin{matrix}\sin\\ \cos\end{matrix}(xu)\,du = g(x), \qquad x > 1 \qquad (2.1.10)$$

where $\psi(u)$ is unknown function and f(x) and g(x) are prescribed functions. Tranter [120] obtained the solution of above equations taking $\alpha = \pm\frac{1}{2}$. These equations are also solved by Seddon [102] by elementary method.

Dwivedi [18] obtained the solution of above equations for general values of α with various possible combinations of trigonometric functions.

Singh and Dhaliwal [100] considered the following set of dual integral equations with trigonometric kernels.

$$\int_0^\infty \left[1 - \frac{2\xi(1-\xi\delta)+1-e^{2\xi\delta}}{2\xi\delta \_+ \sinh 2\xi\delta}\right]\xi A(\xi)\cos(\xi x)\,d\xi = f(x), \qquad 0 < x < a \qquad (2.1.11)$$

$$\int_0^\infty A(\xi)\cos\xi x\,d\xi = 0, \qquad x > a \qquad (2.1.12)$$

(iii) Dual Integral Equations Involving Inverse Mellin Transforms

Srivastav and Parihar [115] considered the following set of dual integral equations involving inverse Mellin transforms:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\psi(s)\rho^{-s}ds = f_1(\rho), \qquad 0<\rho<a \tag{2.1.13}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tan\alpha\psi(s)\rho^{-s}ds = f_2(\rho), \qquad a<\rho<\infty \tag{2.1.14}$$

Erdélyi [36] solved the following set of dual equations

$$M^{-1}\left[\frac{\Gamma(1+\eta-a/\sigma)}{\Gamma(1+\eta+\alpha-a/\sigma)}\phi(s);x\right]=f(x), \qquad 0<x<a \tag{2.1.15}$$

$$M^{-1}\left[\frac{\Gamma(\xi+a/\delta)}{\Gamma(\xi+\beta+a/\delta)}\phi(s);x\right]=g(x), \qquad a<x<\infty \tag{2.1.16}$$

Cooke obtained the solution of the above equations with f(x) = h(x) = 0 and later extended the method for f(x) ≠ 0.

Tweed [124] solved the dual integral equations involving the inverse of certain Mellin type transforms. Recently Trivedi and Pandey [122] have also obtained the solution of dual integral equation involving Naylor's Mellin type transforms by employing Tweed's method of solution but the equations considered by them were quite different from those investigated by Tweed himself. Later on Trivedi and Pandey [123] also solved the two pairs of dual integral equations involving the inverse of Naylor's Mellin type transforms.

(iv) Dual Integral Equations with H-Functions

Dual integral equations involving H-functions as kernels were first introduced by Fox [41]. Later Saxena [91] and Saxena and Kumbhat [94] extended the result obtained by Fox to more general dual equations.

Recently Mehra and Ahuja [60] considered the following set of integral equations:

$$r\int_0^\infty H(x_r, y_r) f(y_r) \prod_{i=1}^{r} (dy_i) = U(x_r), \qquad 0 \le (x_r) < 1 \qquad (2.1.17)$$

where the kernel $H(x_r, y_r)$ is called a multivariable H-function. The second integral equation, defined with other boundary conditions, $(x_r) > 1$, has another H-function as kernel with parameters different from that given above and involves another function $V(x_r)$ on the right hand side while the prefix (r) before the infinite integral sign shows presence of r such operations, the first boundary condition $0 \le (x_r) \le 1$ stands for two sided inequalities $0 \le x_1 < 1$, $0 \le x_2 < 1, \ldots, 0 \le x_r < 1$.

The method used for solution is Fox's method involving the Laplace's operators L and L^{-1} to solve very general integral equation.

(v) Dual Integral Equations Involving Legendre Function

Dhaliwal and Singh [17] considered the following pair of dual integral equations and obtained a closed form solution.

$$\int_0^\infty \tau^{-1} A(\tau) P_{-(1/2)+i\tau}(\cosh\alpha) \tanh(\tau) d\tau = f(\alpha) \qquad 0 < \alpha < a, \qquad (2.1.18)$$

$$\int_0^\infty A(\tau) P_{-(1/2)+i\tau}(\cosh\alpha) d\tau = 0 \qquad a < \alpha, \qquad (2.1.19)$$

where $A(\tau)$ is to be determined, $P_{-(1/2)+i\tau}(\cosh\alpha)$ is a Legendre function of complex index, f is a real positive constant and $f(\alpha)$ is a prescribed function.

Recently Mandal [59] considered certain dual integral equations involving generalized associated Legendre functions of the first kind and trigonometric functions as kernels. Solution of these are obtained by using properties of the generalized associated Legendre functions and the inversion formula for the generalized Mehler-Fok transform involving generalized associated Legendre functions of the first kind.

(vi) Dual Integral Equations Involving Hankel Kernel of Order Zero

Tranter [119] considered the dual integral equations involving Hankel kernel of order zero and solved them, which were not derivable by Titchmarsh and Busbridge's solution. Those equations are

$$\int_0^\infty u^\alpha \{1 + H(u)\} \psi(u) J_0(xu) du = f(x), \qquad 0 < x < 1 \qquad (2.1.20)$$

$$\int_0^\infty \psi(u) J_0(xu) du = 0, \qquad x > 1 \qquad (2.1.21)$$

Labedev and Uflayand [49] obtained the solution in the form of Fredholm integral equation with symmetric kernel of the equations

$$\int_0^\infty \{1 - H(t)\}\phi(t) J_0(xt)dt = f(x), \qquad 0 < x < a \qquad (2.1.22)$$

$$\int_0^\infty \phi(t) J_0(xt) = o, \qquad x < a \qquad (2.1.23)$$

Later on Nasim [67] has shown that the dual integral equations with Hankel kernel

$$\int_0^\infty t^{-2\alpha} J_v(xt)[1 + \omega(t)]\phi(t)dt = f(x), \qquad 0 < x < 1 \qquad (2.1.24)$$

$$\int_0^\infty t^{-2\beta} J_\mu(xt)\phi(t)dt = g(x), \qquad x > 1 \qquad (2.1.25)$$

where ω is an arbitrary weight function, can be reduced to a singular Fredholm integral equation of the first kind.

Mehra and Ahuja [60] solved fractional integrals and a Hankel type transformation on certain spaces of test functions and of distributions. They have shown that the classical relations between fractional integrals and the Hankel transformation persist of these spaces, and thus a classical solution of certain dual integral equations also persist.

(vii) Dual Integral Equations Involving Fourier Transforms

Nguyen and Popov [70] studied dual integral equations on a system of intervals

$$\frac{d}{dx}\int_0^\infty A(\xi)\sin(\xi x)\,d\xi = f_n(x), \qquad x \in I_n, n = 1, \ldots\ldots, N \tag{2.1.26}$$

$$\int_0^\infty A(\xi)\cos(\xi x)\,d\xi = 0, x \in R^+ \setminus \bigcup_{n=0}^{N} I_n \tag{2.1.27}$$

where $A(\xi)$ is an unknown function, f_n are given functions and $I_n = (a_n, b_n)$ are certain finite non-intersecting intervals contained in R^+, where $a_1 < a_2 < \ldots a_n$.

Authors reduced the dual integral equations into an equivalent system of integral equation of the first kind and they gave the criteria for it to have a unique solution. This solution can be obtained by the method of successive approximations. The authors gave the complete mathematical justification of the formal constructions used and constructions in the appropriate function spaces. Recently, Nguyen [71] investigated very elaborately the existence and uniqueness of the problems for certain dual integral equations involving the Fourier transforms of generalized functions. The following dual integral equations are considered

$$PF^{-1}[K(t)\tilde{u}(t)](x) = f(x), \qquad \text{for } x = \omega, \tag{2.1.28}$$

$$P'F^{-1}[\tilde{u}(t)](x) = g(x), \qquad \text{for } x = \omega', \tag{2.1.29}$$

where $\omega' = R/\omega$, $u = F^{-1}[\tilde{u}(t)] \in s'$ is a function to be determined, K(t) is the non-negative function, $f \in D'(\in \omega')$ and $g \in D'(\omega')$ are given distributions on ω and ω' respectively and P, P′ are restriction operators to ω and ω' respectively.

(viii) Dual Integral Equations of Distributional Watson Transforms Type

Pathak [83] extended the reciprocal formula f = g ^ h for the Watson transform of the function f, defined by $g = f \cap k$ on the generalized function space M′ a, b provided the functions k(x) and h(x) satisfy the functional equation K(s) H(1-s) = 1, where ^ denotes the Mellin convolution and K(s) and H(s) are Mellin transforms of k(x) and h(x) respectively. Finally he applied this theory to obtain solution of a pair of dual integral equations.

(ix) Dual Integral Equations of Weber-orr Transform Type

Recently Nasim [68] solved some dual integral equations involving Waber-orr transforms

$$W^{-1}_{v-k,v}\left[\xi^{-2\alpha}\overline{\psi}(\xi),\rho\right] = f_1(\rho), \qquad a \le \rho \le c \qquad (2.1.30)$$

$$W^{-1}_{v-k,v}\left[\xi^{-2\beta}\overline{\psi}(\xi),\rho\right] = -f_2(\rho), \qquad c < \rho < \infty \qquad (2.1.31)$$

where k = 1,2,3, ..., $v > -1$ and $\overline{\psi}$ is an unknown function. Elementary methods are used to construct a general solution that contains many known results.

(x) Dual Integral Equations of Cauchy, Abel and Titchmarsh Type

Estrada and Kanwal [38] considered the following singular dual integral equations and gave distributional solutions.

Cauchy type

$$\alpha_1(x)f(x)+\beta_1(x)\int_{a_1}^{b_1}\frac{f(t)}{(t-x)}dt+\gamma_1(x)\int_{a_2}^{b_2}\frac{f(t)}{(t-x)}dt=g(x), \quad \text{for } a_1 < x < b_1 \tag{2.1.32}$$

$$\alpha_2(x)f(x)+\beta_2(x)\int_{a_1}^{b_1}\frac{f(t)}{(t-x)}dt-\gamma_2(x)\int_{a_2}^{b_2}\frac{f(t)}{(t-x)}dt=g(x), \quad \text{for } a_2 < x < b_2 \tag{2.1.33}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\beta_1(x)$, $\beta_2(x)$, $\gamma_1(x)$, $\gamma_2(x)$ and $g(x)$ are known distributions and $f(t)$ is to be determined.

Abel Type

$$\alpha_1(x)\int_{a_1}^{x}\frac{f(t)}{(t-x)^{\alpha_1}}dt+\beta_1(x)\int_{x}^{b_2}\frac{f(t)}{(t-x)^{\alpha_1}}dt+\gamma_1(x)\int_{a_2}^{b_2}\frac{f(t)}{(t-x)^{\alpha_2}}dt = g(x), \quad \text{for } a_1 < x < b_1 \tag{2.1.34}$$

$$\alpha_2(x)\int_{a_2}^{x}\frac{f(t)}{(t-x)^{\alpha_2}}dt+\beta_2(x)\int_{x}^{b_2}\frac{f(t)}{(t-x)^{\alpha_2}}dt+\gamma_2(x)\int_{a_1}^{b_1}\frac{f(t)}{(t-x)^{\alpha_1}}dt = g(x), \quad \text{for } a_2 < x < b_2 \tag{2.1.35}$$

Titchmarsh Type

$$\int_0^{\infty} t^{-2a} f(t) J_m(tx)\,dt = A(x), \qquad \text{for } 0 < x < 1 \tag{2.1.36}$$

$$\int_0^{\infty} t^{-2b} f(t) J_n(tx)\,dt = B(x), \qquad \text{for } 1 < x < \infty \tag{2.1.37}$$

where the $J_n(tx)$ and $J_m(tx)$ are Bessel functions of order n and m. m, $n \geq -\frac{1}{2}$; a and b are real constants; A(x) and B(x) are prescribed functions.

Recently Eswaran obtained the solution of dual integral equations that are classical for diffraction theory

$$\int_{-\infty}^{\infty} \sqrt{u^4 - k^2}\, A(u) e^{iux}\,du = f(x), \qquad \text{for} |x| < 1 \tag{2.1.38}$$

$$\int_{-\infty}^{\infty} A(u) e^{iux}\,du = 0, \qquad \text{for} |x| > 1 \tag{2.1.39}$$

by reducing them to an infinite system of linear algebraic equations.

After that more general pair of the form

$$\int_{-\infty}^{\infty} F\, \nu_1, \nu_2, \ldots\ldots, u A(u) e^{iux}\,du = f(x), \qquad \text{for} |x| < 1 \tag{2.1.40}$$

$$\int_{-\infty}^{\infty} A(u) e^{iux}\,du = 0, \qquad \text{for} |x| > 1 \tag{2.1.41}$$

where F is some given function with a number of specified properties.

$$\nu_{1,2}(u) = \sqrt{\left(u^2 - k_{1,2}^2\right)}, k_{1,2} \geq 0, \ \ k_1 + k_2 > 0$$

undergoes a similar treatment.

Prabha [87] obtained a formal solution of dual integral equations of two variables with two weight functions using the fractional integral operators by reducing them to Fredholm integral equations of the second kind.

Veliev and Shestopalov [127] considered the following set of dual integral equations and found a general method to solve them.

$$\int_{-\infty}^{\infty} h(\alpha)k(\alpha)e^{\pm i\xi\alpha\eta}d\alpha = f(\eta), \qquad |\eta| < 1 \tag{2.1.42}$$

$$\int_{-\infty}^{\infty} h(\alpha)e^{\pm i\xi\alpha\eta}d\alpha = 0, \qquad |\eta| > 1 \tag{2.1.43}$$

These equations are in the class of functions $h(\eta)$ for which $h(\eta)$, $|\eta|^{1/2}h(\eta) \in L_2(k)$ under certain assumptions on $k(\eta)$ and $f(\eta)$. They proved the unique solvability of this equation.

2.1.2 Simultaneous Dual Integral Equations

Erdogen and Bahar [37] obtained for the first time the solution of the simultaneous dual integral equations involving Bessel functions.

These equations are

$$\int_0^\infty \sum_{n=1}^\infty C_{ij}(x)\psi_j(x)J_{\mu_i}(xy)dx = P_i(y), \qquad y \in I_1 \qquad (2.1.44)$$

$$\int_0^\infty \psi_j(x)\; J_{\mu_i}(xy)dx = 0, \qquad y \in I_2 \qquad (2.1.45)$$

where I = 1,2, …., n. A particular case of above equations for n = 2 is solved by Westman [238]. Dwivedi [40] and Dwivedi and Singh [49] also considered some other sets of simultaneous dual integral equations.

Narain and Lal [66] considered simultaneous dual integral equations involving Maijer's G-functions of n-variables.

We have also considered certain simultaneous dual integral equations associated with kernel of Fox and simultaneous dual integral equations of convolution type in the present thesis.

2.1.3 Triple Integral Equations

(i) Triple Integral Equations Involving Bessel Functions

Tranter [121] was the first person who extended dual integral equations to triple integral equations. He considered the equations:

$$\int_0^\infty \psi(u)J_\nu(xu)du = \begin{cases} f(x), & 0 < x < a \\ h(x), & b < x < \infty \end{cases} \qquad (2.1.46)$$

$$\int_0^\infty u^\alpha \psi(u)\{1+H(u)\}J_\nu(xu)\,du = g(x), \quad a < x < b \tag{2.1.47}$$

Srivastava [112] considered the following two sets of triple integral equations involving Bessel function as kernel.

I Set

$$\int_0^\infty s\,A(s)\,J_\nu(\rho s)\,ds = \begin{cases} 0, & 0 < \rho \le a \\ 0, & 1 \le \rho < \infty \end{cases} \tag{2.1.48}$$

$$\int_0^\infty s\,A(s)\,J_\nu(\rho s)\,ds = -f(\rho), \quad a \le \rho \le 1 \tag{2.1.49}$$

II Set

$$\int_0^\infty \left(\lambda s + \mu s^2\right) A(s)\,J_\nu(\rho s)\,ds = \begin{cases} 0, & 0 < \rho \le a \\ 0, & 1 \le \rho < \infty \end{cases} \tag{2.1.50}$$

$$\int_0^\infty A(s)\,J_\nu(\rho s)\,ds = f(\rho), \qquad a \le \rho \le 1 \tag{2.1.51}$$

The triple integral equations are solved by reducing them to ordinary differential equation and the solution of this ordinary differential equation together with the inversion of Abel integral equation yields Fredholm integral equation of the second kind.

Cooke [14] put the solutions of triple integral equations involving Bessel function on sound footing by making heavy use of Erdélyi-Köber

operators and introduced new operators to obtain the solutions.

(ii) Triple Integral Equations with Trigonometric Kernels

Srivastava and Lowengrub [110] solved first time the triple integral equations with trigonometric kernel by applying finite Hilbert transform technique. Later on Singh [96] considered the different set of equations

$$\int_0^\infty A(t)\cos(xt)dt = \begin{cases} 0, & 0 < x < a \\ 0, & x > b \end{cases} \tag{2.1.52}$$

$$\int_0^\infty A(t)\, t \tanh(xt)\cos(xt)dt = p(x), \qquad a < x < b \tag{2.1.53}$$

and solved them.

(iii) Triple Integral Equations Involving Inverse Mellin Transforms

Srivastav and Parihar [115] considered the triple integral equations of the type

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi(s)\rho^{-s}ds = \begin{cases} f_1(\rho), & 0 < \rho < a \\ f_3(\rho), & \rho > 1 \end{cases} \tag{2.1.54}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi(s)\tan\alpha\,\rho^{-s}ds = f_2(\rho), \qquad a < \rho < 1 \tag{2.1.55}$$

They reduced the equations into dual cosine series equations and then wrote the solutions by using Tranter's method.

Lowndes [55] extended the operators of fractional integration, considered earlier by Cooke, and obtained the solution of the following set of triple integral equations.

$$M^{-1}\left[\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\phi(s);x\right]=\begin{cases}0, & 0<x<a\\ f_1(x), & b<x<\infty\end{cases} \tag{2.1.56}$$

$$M^{-1}\left[\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta-\alpha-s/\sigma)}\phi(s);x\right]=f_2(x), \quad a<x<b \tag{2.1.57}$$

where α, β, ξ, η, $\delta > 0$, $\sigma > 0$ are real parameters.

Singh [96] considered triple integral equations of inverse Mellin type. Recently Tweed [125], considered triple integral equations involving finite inverse Mellin transforms. The equations are solved by the method of reducing the triple integral equations to a singular integral equation.

In this thesis we have also considered the two different sets of triple integral equations with Legendre function as kernel and solved them using the properties of generalized Legendre functions and inversion theorem for Mehler – Fock transform.

(iv) Triple Integral Equations with H-functions

Saxena and Kumbhat [95] for the first time solved the triple integral equations involving H-functions. Recently Mehra and Prabha [61] obtained the formal solution of certain triple integral equations containing H-functions of n-variables.

(v) Triple Integral Equations with Legendre Functions of Imaginary Argument As Kernel

The triple integral equations containing Legendre function as kernel were considered by Srivastava [113]. Recently Dwivedi considered more general triple equations and gave better solution than those given by the previous authors.

(iv) Triple Integral Equations Involving Mehler-Fock Transform

Srivastava [113] considered the following two sets of triple integral equations involving Mehler-Fock transforms.

I Set:

$$\int_0^\infty \tau\, m(\tau) A(\tau) p^{\beta}_{-(1/2)+i\tau}(\cosh x)\, d\tau = 0, \qquad 0 \le x \le a \qquad (2.1.58)$$

$$\int_0^\infty \left[\frac{1}{4}+\tau^2\right] A(\tau) p^{\beta}_{-(1/2)+i\tau}(\cosh x)\, d\tau = -f(x), \qquad a \le x \le 1 \qquad (2.1.59)$$

$$\int_0^\infty \tau\, m(\tau) A(\tau) p^{\beta}_{-(1/2)+i\tau}(\cosh x)\, d\tau = 0, \qquad 1 \le x \le \infty \qquad (2.1.60)$$

II Set:

$$\int_0^\infty \tau\, A(\tau) p^{\beta}_{-(1/2)+i\tau}(\cosh x)\, d\tau = 0, \qquad 0 \le x \le a \qquad (2.1.61)$$

$$\int_0^\infty \left[\frac{1}{4}+\tau^2\right] m(\tau)A(\tau)p^{\beta}_{-(1/2)+i\tau}(\cosh x)\,d\tau = -f(x), \quad a \le x \le 1 \qquad (2.1.62)$$

$$\int_0^\infty \tau\, A(\tau)p^{\beta}_{-(1/2)+i\tau}(\cosh x)\,d\tau = 0, \qquad 1 \le x < \infty \qquad (2.1.63)$$

The above triple integral equations are first reduced to the problem of solving an ordinary differential equation. This differential equation is solved together with inversion theorem for some variants of Abel integral equation of the second kind. At last the problems of solving (triple integral equations in reduced to that of solving) Fredholm integral equations of the second kind.

2.1.4 Simultaneous Triple Integral Equations

Dwivedi and Sharma [25] have recently considered the simultaneous triple integral equations involving H and G-functions of two variables. Dwivedi and Singh [24] generalized the problems of H-functions of n-variables. Recently Paliwal and Mishra [76] have also considered the formal solution of the simultaneous triple integral equations, which are very useful for some crack problem in the mathematical theory of elasticity.

2.1.5 Quadruple Integral Equations

(i) Quadruple Integral Equations Involving Bessel Functions

Ahmad [2] has done the extension of triple integral equations to quadruple integral equations and those integral equations are as follows:

$$\int_0^\infty u^{-2\alpha}\phi(t)J_\nu(xt)\,dt=\begin{cases}F_1(x), & 0<x<a\\ F_3(x), & b<x<c\end{cases} \tag{2.1.64}$$

$$\int_0^\infty u^{-2\beta}\phi(t)J_\nu(xt)\,dt=\begin{cases}F_2(x), & a<x<b\\ F_4(x), & x>c\end{cases} \tag{2.1.65}$$

He solved the above set of integral equations by suing Erdélyi-Köber operators and reduced them in the form of Fredholm integral equations of the second kind.

Cooke [15] solved the above equations with general weight function by considered some more general operators. Dwivedi, Trivedi and Kushwaha [26] generalized the results of Cooke [15]. Gupta and Chaturvedi [45], Saxena and Sethi [93] have also solved quadruple integral equations with Bessel function kernels.

Recently Prabha [86] considered cretain quadruple integral equations involving Bessel functions as kernels and solved them by the application of generalized operators of the Hankel transform by Erdélyi-Köber operators of two variables.

(ii) Quadruple Integral Equations Involving Trigonometric Kernel

Singh and Jain [99] considered quadruple integral equations involving trigonometric kernels, which are as follows:

$$\int_0^\infty t\,A(t)\cot(\alpha t)\,dt=\begin{cases}f_1(\alpha), & 0<\alpha<a\\ f_3(\alpha), & b<\alpha<c\end{cases} \tag{2.1.66}$$

$$\int_0^{\infty} A(t)\cos(\alpha t)\coth(\pi t)\,dt = \begin{cases} f_2(\alpha), & a < \alpha < b \\ 0 & c < \alpha < \infty \end{cases} \tag{2.1.67}$$

The above set of integral equations arises in two dimensional steady state heat conduction problems. Singh [98] considered integral equations and its applications to electrostatics.

(iii) Quadruple Integral Equations Involving Inverse Mellin Transform

Recently Dwivedi, Kushwaha and Trivedi [26] have considered the following set of quadruple integral equations involving inverse Mellin transform:

$$M^{-1}\left\{\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\phi(s); x\right\} = \begin{cases} f_1(x), & 0 \le x < a \\ f_3(x), & b < x < c \end{cases} \tag{2.1.68}$$

$$M^{-1}\left\{\frac{\Gamma(\xi+s/\sigma)}{\Gamma(\xi+\beta+s/\delta)}\phi(s); x\right\} = \begin{cases} g_2(x), & a < x < b \\ g_4(x), & c < x < \infty \end{cases} \tag{2.1.69}$$

Other authors also considered certain quadruple integral eqautions of inverse Mellin transform type (see bibliography).

(iv) Quadruple Integral Equations with H-Function

Saxena and Sethi [93] introduced us the Quadruple integral equations with H-functions as kernel. Such quadruple integral equations are the extensions of the dual and triple integral equations involving same kernel.

(v) Quadruple Integral Equations with Legendre Functions of Imaginary Argument as Kernal

Dange and Singh [16] extended the paper of Srivastava [108] for triple integral equations with Legendre functions of imaginary argument as kernel to the following set of quadruple integral equations

$$\int_0^\infty A(V)\, p_{-(1/2)+iv}(\cos h\alpha)\, dV = \begin{cases} f_1(\alpha), & 0<\alpha<a \\ f_3(\alpha), & b<\alpha<c \end{cases} \qquad (2.1.70)$$

$$\int_0^\infty V\tanh(\pi V) A(V)\, p_{-(1/2)+iv}(\cos h\alpha)\, dV = \begin{cases} f_2(\alpha), & a<\alpha<b \\ f_4(\alpha), & c<\alpha<\infty \end{cases} \qquad (2.1.71)$$

where A(V) is to be determined.

2.1.6 5-Tuple Integral Equations

None of the authors havė written a paper on five integral equations involving kernel of any kind till date. Dwivedi and Singh [33] have derived the solution of n-integral equations from where the solution of five integral equations can be obtained as a particular case.

2.1.7 6-Tuple Integral Equations

(i) Six Integral Equations Involving Bessel Functions

Ahuja [3] considered following set of six integral equations involving Bessel functions as kernel

$$\int_0^\infty \xi^p \psi(\xi) J_\nu(\xi x) d\xi = \begin{cases} f_1(x), & 0 < x < a_1 \\ f_3(x), & a_2 < x < a_3 \\ f_5(x), & a_4 < x < a_5 \end{cases} \tag{2.1.72}$$

$$\int_0^\infty \xi^{-q} \psi(\xi) J_\nu(\xi x) d\xi = \begin{cases} g_2(x), & a_1 < x < a_2 \\ g_4(x), & a_3 < x < a_4 \\ g_6(x), & a_5 < x < \infty \end{cases} \tag{2.1.73}$$

where ξ^p and ξ^q are weight functions, $\pm p \neq \pm q$, the values of p and q are taken to be 0, 1, -1. The solution of this set is in the form of simultaneous Fredholm integral equations.

(ii) Six Integral Equations Involving Legendre Functions

Dwivedi, Kushwaha and Gupta [28], obtained the solution of following set of six integral equations involving Legendre functions

$$\int_0^\infty \phi(\tau)\, p^{\beta}_{-(1/2)+i\tau}(\cos h\alpha)\, d\tau = \begin{cases} f_1(\alpha), & 0 < \alpha < a_1 \\ f_3(\alpha), & a_2 < \alpha < a_3 \\ f_5(\alpha), & a_4 < \alpha < a_5 \end{cases} \tag{2.1.74}$$

$$\int_0^\infty \frac{\tau}{\pi} \sinh(\pi\tau) \Gamma\left(\frac{1}{2} - \beta + i\tau\right)\left(\frac{1}{2} + \beta - i\tau\right) p^{\beta}_{-(1/2)+i\tau}(\cos h\alpha)\, \alpha\, \phi(\tau) d\tau$$

$$= \begin{cases} g_2(\alpha), & a_1 < \alpha < a_2 \\ g_4(\alpha), & a_3 < \alpha < a_4 \\ g_6(\alpha), & a_5 < \alpha < a_6 \end{cases} \tag{2.1.75}$$

where $f_i(\alpha)$, $g_i(\alpha)$, i = 1,3,5; j = 2,4,6 are the prescribed functions and $\phi(\tau)$ is an unknown, to be determined. The final solution is in the form of a Fredholm integral equation.

Dwivedi and Gupta [29] also considered six integral equations with generalized Legendre functions as kernel. Those equations are as follows:

$$\int_0^\infty \psi(\tau)\, p^{\mu}_{-(1/2)+i\tau}(\cos hx)\,d\tau = f_i(\alpha),\ a_{j-i} < \alpha < a_j, \qquad j = 1,3,5 \text{ and } a_0 = 0 \tag{2.1.76}$$

$$\int_0^\infty \psi(\tau)\,\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\tau \sinh \pi\tau\, p^{-\mu}_{-(1/2)+i\tau}(\cosh\alpha)\,d\tau = g_j(\alpha),\ a_{j-1} < \alpha < a_j,\ j = 1,4,6 \text{ and } a_6 = \infty \tag{2.1.77}$$

where f_j and g_j are prescribed functions and $\psi(\tau)$ is to be determined. The final solution is in the form of Fredholm integral equation of the second kind.

2.1.8 n-Integral Equations

(i) n-Integral Equations Involving Bessel Functions

No author has written a paper on n-integral equations involving kernel of any kind. Recently Dwivedi and Singh [33] considered n-integral equations involving Bessel functions and obtained the solution by reducing them to Fredholm integral equations of second kind.

2.2 SERIES EQUATIONS

2.2.1 Dual Series Equations

(i) Dual Series Equations Involving Bessel Functions

For the first time Cooke and Tranter [12] studied the following set of

dual series equations involving Bessel functions

$$\sum_{n=0}^{\infty} \lambda_n^{-2P} A_n J_v(\rho\lambda_n) = F(\rho), \qquad 0 \le \rho \le 1, \tag{2.2.1}$$

$$\sum_{n=0}^{\infty} A_n J_v(\rho\lambda_n) = 0, \qquad 1 < \rho < a, \tag{2.2.2}$$

They reduced the above equations to a system of algebraic equations which can be easily solved by numerical methods.

Sneddon and Srivastav [103] considered many other types of series equations.

Srivastav [212] introduced the equations

$$A_0 + \sum_{n=1}^{\infty} \lambda_n^{2P} A_n J_v(\rho\lambda_n) = F(\rho), \qquad 0 \le \rho \le c, \tag{2.2.3}$$

$$\sum_{n=1}^{\infty} A_n J_v(\rho\lambda_n) = G(\rho), \qquad c < \rho \le 1 \tag{2.2.4}$$

where $\{\lambda_n\}$ is the sequence of positive roots of the transcendental equation

$$\lambda J_v(\lambda) + \mu J_v(\lambda) = 0$$

H and ν being the real coefficients with $\nu = -1/2$ and $\mu + \nu \ge 0$.

(ii) Dual Series Equations of Jacobi Polynomials

Noble [72] considered the following set of dual series equations:

$$\sum_{n=0}^{\infty} P_n(\nu,\beta) A_n J_n(\alpha,\beta,x) = F(x), \qquad 0 \le x < a \tag{2.2.5}$$

$$\sum_{n=0}^{\infty} A_n J_n(\alpha,\beta,x) = g(x), \qquad a < x < 1 \tag{2.2.6}$$

where $J_n(\alpha, \beta, x)$ denotes the Jacobi polynomial and $P_n(\nu,\beta)$ is a constant defined by

$$P_n(\nu,\beta) = \frac{\Gamma(\nu+n)\Gamma(1+\alpha-\beta+n)}{\Gamma(1+\alpha-\nu+n)\Gamma(\beta+n)} \tag{2.2.7}$$

He solved the above equations by developing new method of multiplying factor.

Srivastav [114] considered the following set of dual series equations

$$\sum_{n=0}^{\infty} \frac{A_n P_n(\alpha,\beta)(\cos\theta)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+3/2)} = F(\theta), \qquad 0 \le \theta \le \phi \tag{2.2.8}$$

$$\sum_{n=0}^{\infty} \frac{A_n P_n(\alpha,\beta)(\cos\theta)}{\Gamma(\beta+n+1)\Gamma(\alpha+n+1/2)} = G(\theta), \qquad \phi < \theta \tag{2.2.9}$$

where $\alpha > -1/2$, $\beta = -1$ and $P_n(\alpha,\beta)(\cos\theta)$ denotes the Jacobi polynomials. He obtained closed form solution by using Abel integral equation method.

(iii) Trigonometrical Dual Series Equations

Tranter [120] considered many different kinds of series equations and solved them. After some time Tranter [120] gave an exact solution of

trigonometric series equations. Noble and whiteman [74] also considered other set of trigonometric series equations.

(iv) Dual Series Equations Involving Generalized Laguerre Polynomials

Srivastava [106] considered following set of dual series equations and solved them by applying Sneddon's [102] method

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{\nu}(x) = f(x), \qquad 0 \leq x < y \tag{2.2.10}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{\alpha}(x) = g(x), \qquad y \leq x < \infty \tag{2.2.11}$$

(v) Dual Series Equations Involving Legendre Polynomials

Collins [10] considered the following set of dual series equations

$$\sum_{n=0}^{\infty} (1+H_n) A_n T_{m+n}^{-m}(\cos\theta) = F(\theta), \qquad 0 \leq \theta < \phi \tag{2.2.12}$$

$$\sum_{n=0}^{\infty} (2n+2m+1) A_n T_{m+n}^{-m}(\cos\theta) = G(\theta), \qquad \phi \leq \theta < \pi \tag{2.2.13}$$

where $T_n^{-m}(\cos\theta)$ is the associated Legendre polynomial. He found the solution of the above equations in the form of Fredholm Integral equations of second kind.

Pathak [82] considered the equations

$$\sum_{n=0}^{\infty} \frac{A_n T^{-n} \rho^n}{\Gamma\left(\nu + \frac{1}{2} + n + \rho\right)} P_{n+\rho,\nu}(x,-t) = f(x,t), \qquad 0 \le x < y \qquad (2.2.14)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma\left(\mu + \frac{1}{2} + n + \rho\right)} P_{n+\rho,\sigma}(x,-t) = g(x,t), \qquad y < x < \infty \qquad (2.2.15)$$

where $P_n\ (x,t)$ is a heat polynomial and solution is obtained in closed form.

Patil [84] studied the dual series equations

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\delta + 1 + K_n)} Z_n^{\delta}(x,k) = f(x), \qquad 0 \le x < y \qquad (2.2.16)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\delta + \beta + K_n)} Z_n^{\sigma}(x,k) = g(x), \qquad y < x < \infty \qquad (2.2.17)$$

where $Z_n^{\alpha}(x,k)$ is the Konhouser biorthogonal polynomial.

(vi) Dual Series Equations with Generalized Bateman K-Functions

Srivastava [107] considered the following set of dual series equations

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta + \sigma + n + 1)} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = f(x), \qquad 0 \le x < y \qquad (2.2.18)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\nu + \sigma + n + 1)} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = g(x), \qquad y \le x < \infty \qquad (2.2.19)$$

where $K_{\nu}^{\alpha}(x)$ is the generalized Bateman K-function. Srivastava [107] also

obtained the solution of particular cases of the above equations.

2.2.2 Simultaneous Dual Series Equations

Dwivedi et al. [31] solved the simultaneous dual series equations involving the product of r-Laguerre polynomial. Dwivedi [21] also considered the simultaneous dual equations involving H-functions.

Narain and Lal [65] obtained the solution of simultaneous dual series equations involving generalized Bateman K-functions. Recently Dwivedi and Shukla [31] considered the simultaneous dual series equations.

2.2.3 Triple Series Equations

First time triple series equations were solved the Collins [11]. He solved the following set of equations

$$\sum_{n=0}^{\infty}(2n+1)C_n P_n(\cos\theta)=0,\ 0<\theta<\alpha,\ \beta<\theta<\pi \tag{2.2.20}$$

$$\sum_{n=0}^{\infty}(1+H_n)C_n P_n(\cos\theta)=f(\theta),\ \alpha<\theta<\beta \tag{2.2.21}$$

which are the equations of the first kind and

$$\sum_{n=0}^{\infty}(1+H_n)C_n P_n(\cos\theta)=f(\theta),\ 0<\theta<\alpha,\ \beta<\theta<\pi \tag{2.2.22}$$

$$\sum_{n=0}^{\infty}(2n+1)C_n P_n(\cos\theta)=0,\ \alpha<\theta<\beta \tag{2.2.23}$$

are the equations of the second kind. Srivastava [109] considered triple series equations involving series of Jacobi polynomials.

Lowndes [56], Dwivedi and Gupta [30], and Dwivedi, Gupta and Shukla [31] obtained the solution of certain triple series equations involving Jacobi polynomials.

Melrose and Tweed [62] obtained the solution of some triple trigonometrical series. Parihar [78] obtained the closed form solution of some triple trigonometrical series. Lowndes and Srivastava [56] have considered a class of triple series equations with Laguerre polynomial kernels and reduced them to triple integral equations with Bessel function kernels. The operators in the integral equations are for modified Hankel transformations. Hence, identities which connect Erdélyi-Köber fractional integral operators with Hankel operator are used to reduce the problem to a single integral equation involving a modified Hankel operator. Inversion then leads to an exact solution, and thus an exact solution to the series equations can be obtained.

In chapter five of the present thesis we have given a solution of a new class of triple series equations with associated Legendre functions which are more general than considered as yet some particular cases have also been discussed.

2.2.4 Quadruple Series Equations

First time Cooke [15] obtained the solution of following quadruple series equations:

$$\sum_{n=0}^{\infty} a_n \lambda_n^{-2\alpha} J_v(\lambda_n x) = \begin{cases} G_2(x), & a < x < b \\ 0, & c < x < d \end{cases} \tag{2.2.24}$$

$$\sum_{n=0}^{\infty} a_n \lambda_n^{-2\alpha} J_v(\lambda_n x) = \begin{cases} F_1(x), & 0 < x < a \\ F_3(x), & b < x < c \end{cases} \tag{2.2.25}$$

Dwivedi and Trivedi [22] considered some quadruple series equations involving Jacobi polynomials.

2.2.5 5-Tuple Series Equations

Dwivedi et al. [32] obtained the solution of various set of five series equations involving Jacobi and Laguerre polynomials; and generalized Bateman K-functions.

In the present thesis we have obtained the solution of five series equations involving Jacobi polynomials which are still untouched till date.

2.2.6 7-Tuple Series Equations

In Chapter Seven of the present thesis we have considered different sets of seven series equations involving generalized Bateman K-functions and generalized Laguerre polynomials.

2.3 APPLICATION OF INTEGRAL EQUATIONS IN THE FIELD OF ELASTICITY

It has been seen that integral and series equations are very useful in the theory of elasticity, electrostatics, elastostatics, diffraction theory and

acoustics. Particularly, these equations are very much useful in finding the solution of crack problems fracture mechanics.

First attempt for theory of cracks was done by Inglis [46] in the first quarter of the 20^{th} century in which he presented the solution of a problem within the classical theory of elasticity concerning the equilibrium of an infinite body with an isolated cavity. After that Muskhelishvili [64] in 1919 solved the same problem in much simpler way.

Griffith presented his first paper [43] in 1920 and second paper [44] in 1925 in which he rightly considered fundamentals for the theory of cracks.

Inglis [46] considered the crack in the shape of an elliptical hole. Griffith [44] made the minor axis of the ellipse to be zero and reduced the crack into a straight line, which is now known as Griffith crack.

Sneddon [104] developed dual integral equations and applied first transform techniques in solving the variety of mixed boundary value problems of the potential theory. Muskelishvili [64] developed a method based on the theory of Cauchy's integral for solving the mixed boundary value problems. These methods have played an important role in the development of potential theory and mathematical theory of elasticity. Sneddon [104] has dealt with different methods to solve mixed boundary value problems.

Recently Tweed and Melrose [126] also considered 'The out of plane shear problem for an infinite sheet with a staggered array of pairs of cracks. They used a triple series technique to find closed form expressions for the

mode III stress intensity factors of a staggered array of pairs of cracks in an infinite elastic solid.

A 2-dimensional problem of an anisotropic elastic strip having an infinite row of Griffith cracks was considered by Misra and Misra [63] by using dual integral equations approach. The problem analyzing analytically the stress intensity factor, the critical pressure and the energy required to open the crack, were studied. Numerical results were also derived.

Griffith's 2-dimensional theory of rupture was extended to 3-dimensional by Sack [90]. He considered a disk shaped crack in the interior of infinite elastic solid. It is usually known as "penny shaped".

Lal and Jain [51] studied the problem of finding the distribution of stress and the displacement in an elastic half plane containing an external line crack perpendicular to the free surface of the plane. This problem has been formulated in the form of dual integral equations. Finally, the expressions for the stress intensity factor and the crack energy are derived numerically.

Palaiya and Majumdar [75] solved the problem of 2-diffraction of harmonic antiplane shear waves by a pair of coplaner parallel rigid strips located at the interfaces by using Fourier cosine transforms. The problem was reduced to triple integral equations, whose solution was obtained by usual methods.

Srivastava and Dhawan [111] considered the problem of stress distrubition due to a Griffith crack at the interface of an elastic layer bonded

to half plane. They reduced this problem to a system of simultaneous dual integral equations involving trigonometric kernels. Some further crack problems have been solved by using simultaneous dual integral equations.

Lowengrub and Sneddon [54] considered the problem of stress distribution in a dissimilar medium when a penny-shaped crack is situated along the bounding plane. They converted this problem to simultaneous dual integral equations which were later solved by known methods.

Certain triple integral equations have been found very useful in the solution of three part boundary value problems. Lowengrub and Srivastava [52] considered certain problems where such equations arise. Several other research workers used triple integral equations to solve different crack problems.

Lowengrub [53] considered the problem of stress distribution for the two bounded dissimilar elastic half planes containing a pair of coplaner cracks at the interface line by using Fourier transforms. He reduced the problems to simultaneous triple integral equations and later to a Hilbert-problem.

Roy and Chatterjee [89] considered the effect of the free surface of the stress distribution of an elliptic crack aligned parallel to the free boundary and at depth h-below it. The title problem is posed to dual integral equations in Cartesian co-ordinate system. By suitable transformation the dual integral equations are first reduced to an infinite system of dual integral in cylindrical

coordinates. Then they are further reduced by a recently developed technique to an infinite system of Fredholm integral equations of the second kind.

The problem of determining the stress intensity factors in a semi-infinite orthotropic elastic medium containing two coplaner cracks parallel to the boundary was considered by Mahapatra and Parti [57]. The above mixed boundary value problem was reduced to triple integral equations which were solved by using infinite Hilbert-transform technique.

Quadruple integral equations have also been used to find the solution of certain crack problems. Singh [97] considered the problem of determining the stress distribution in the vicinity of a Griffith crack in an infinite elastic solid, when the crack is opened by two symmetrical rigid inclusions.

Recently, Lal [50] considered the problem of elastic half-space under torsion by a flat annular rigid stamp in the linear micropolar elasticity. The problem is reduced to system of four Fredholm integral equations.

2.4 APPLICATIONS OF SERIES EQUATIONS IN CRACK PROBLEMS OF ELASTICITY

A large number of problems are encountered in the theory of crack problems of elasticity. Some of these problems are being mentioned below where series equations have been found useful.

Parihar and Garg [79] derived the stress and displacement distributions when infinite Griffith cracks are presented along the bound line of two

dissimilar elastic half planes. The problem is first reduced to the dual series equations and then to a Hilbert problem.

Parihar and Kushwaha [80] considered the problem of distribution of stress in an infinite strip containing two Griffith cracks under the action of body forces. They reduced the problem to a set of triple series equations and obtained the closed form solution.

Parihar and Kushwaha [81] also considered a crack problem of a strip containing Baranbalatt crack. They reduced the problem to dual series equations. By symmetry conditions the problem is equivalent to that of an infinite row of collinear Baranbalatt cracks in an infinite elastic solid.

The problem of stress distribution in the presence of infinite row of interface cracks, located symmetrically on the central line in a composite strip was considered by Parihar and Garg [79]. This problem is first converted to simultaneous dual series equations and finally to Hilbert problem.

A number of crack problems in the classical theory of elasticity have been solved with the help of integral and series equations. A complete bibliography of the work done till 1967 can be found in a monograph by Lowengrub and Sneddon [54].

In the present thesis, we have used triple integral equations to solve a crack problem of elasticity.

CHAPTER – 3

SIMULTANEOUS DUAL INTEGRAL EQUATIONS

In this chapter we shall discuss solutions of certain simultaneous dual integral equations. In sections 3.1 and 3.2, the simultaneous dual equations involving H-functions and convolution type have been considered respectively.

3.1 CERTAIN SIMULTANEOUS DUAL INTEGRAL EQUATIONS ASSOCIATED WITH KERNEL OF FOX

In this section the formal solution of certain simultaneous dual integral equations involving H-functions is obtained by the method of fractional integration. By the application of fractional integration operators, the given simultaneous equations are transformed into two others having common kernel and the problem then reduces to that of solving one integral equation. Since the common kernel obtained is a symmetrical Fourier kernel given earlier by Fox the solution then follows easily.

3.1.1 Introduction

Fox [41] and Saxena [91], [92], [94] have obtained formal solution of certain dual integral equations involving H-functions. In this section we have discussed certain simultaneous dual integral equations of more general nature associated with the kernel of Fox [41]. By the application of fractional integration operators the given integral equations are transformed into two

others with common kernel and problem then reduced to that of solving one integral equation. Since the common kernel obtained comes out to be a symmetrical Fourier kernel investigated earlier by Fox, the formal solution is readily obtained.

We define the H-function by means of Mellin Barnes integral [91], in the form

$$H_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_p,A_p)\\(b_p,B_q)\end{matrix}\right]=H_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_1,A_1)......(a_p,A_p)\\(b_1,B_1)......(b_p,B_q)\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L\frac{\overset{n}{\underset{j=1}{\pi}}\Gamma(b_j+b_js)\overset{n}{\underset{r=1}{\pi}}\Gamma(1-a_r-a_rs)}{\overset{\rho}{\underset{j=m+1}{\pi}}\Gamma(1-b_j-B_js)\overset{\rho}{\underset{r=n+1}{\pi}}\Gamma(a_r+a_rs)}X^{-s}ds \tag{3.1}$$

where x is not equal to zero and an empty product is to be interpreted as unity; p,q,m and n are integers satisfying $1 \le m \le q$, $0 \le n \le 1$; A and B_j are positve numbers and a and b_j are complex numbers (where r = 1, 2,..... p and j = 1, 2,....., q) such that no pole of $\Gamma(b_j + B_js)$ (j = 1,2,3......m) coincides with any pole of $\Gamma(1 - B_r - sA_r)$ (r = 1, 2, 3,.....n) i.e. $A_r(b_r + \nu) \neq B_j(a_r - \eta - 1)$ for $\nu, \eta = 0,1,2,....$

The above definition possesses one advantage that for a suitable contour L, and coefficient of x^{-5} in the integrand of (3.1) is evidently the Mellin transform of $H_{p,q}^{m,n}(x)$.

An account of the covergence conditions of the integrand (2.1) could

be found in the work of Braaksma [5] and Erdelyi, et al. [34].

Fox has shown that the function

$$H^{n,m}_{2m,2n}\left[x\left|\begin{matrix}(1-a_m, A_m)(a_m - A_m, A_m)\\(b_n, B_n),(1-b_n - B_n, B_n)\end{matrix}\right.\right] = H^{n,m}_{2m,2n}(x)$$

$$= \frac{1}{2\pi i}\int_L \chi_{m,n}(s) x^{-5} ds, \tag{3.2}$$

where

$$\chi_{m,n}(s) = \overset{n}{\underset{i=1}{\pi}} \frac{\Gamma(b_i + sB_i)}{\Gamma(b_i + B_i - sB_i)} \overset{m}{\underset{j=1}{\pi}} \frac{\Gamma(a_j - sA_j)}{\Gamma(a_j - A_j + sA_j)} \tag{3.3}$$

behaves as a symmetrical Fourier kernel from (3.2) it follows that Mellin transform of $H^{n,m}_{2m,2n}(x)$ is $\chi_{m,n}(s)$.

3.1.2 Simultaneous Dual Integral Equations

The solution of the following simultaneous dual integral equations will be obtained here:

$$H^{n,m}_{2m,2n}\left[xu\left|\begin{matrix}(1-\alpha^k_m, A_m)(a_m - A_m, A_m)\\(b_n, B_n),(1-\beta_n - B_n, B_n)\end{matrix}\right.\right]\sum_{h=1}^{n} a_{hk} f_h(u) du$$

$$= \phi_k(x), \qquad 0 < x < 1 \tag{3.4}$$

and

$$H^{n,m}_{2m,2n}\left[xu\left|\begin{matrix}(1-a_m,A_m)(\alpha^k_m-A_m,A_m)\\(\beta_n,B_n),(1-b_n-B_n,B_n)\end{matrix}\right.\right]\sum_{h=1}^{n}b_{hk}f_h(u)du$$

$$=\psi_k(x), \qquad x>1 \tag{3.5}$$

where k = 1, 2,....., n; $\phi_k(x)$ and $\psi_k(x)$ are given, $f_h(u)$ are to be determined and a_{hk}, b_{hk} are known constants. The H-functions used here are as defined in (3.2). In above equations (3.4) and (3.5) the unknown function $f_h(u)$ must be bounded and integrable.

If we put M[f(u)] = F(s) and apply Fox's result [91] to (3.4) and (3.5), we obtain

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)x^{-s}\sum_{h=1}^{n}a_{hk}F_h(1-s)ds=\phi_k(x), \qquad 0<x<1 \tag{3.6}$$

and

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)x^{-s}\sum_{h=1}^{n}b_{hk}F_h(1-s)ds=\psi_k(x), \qquad x>1 \tag{3.7}$$

where k = 1, 2,.....n and

$$\chi^*_{m,n}(s)=\prod_{i=1}^{n}\frac{\Gamma(b_i+sB_i)}{\Gamma(\beta^k_i+B_i-sB_i)}\prod_{j=1}^{m}\frac{\Gamma(\alpha^k_j-sA_j)}{\Gamma(a_j-A_j+sA_j)} \tag{3.8}$$

$$\chi^{**}_{m,n}(s) = \prod_{i=1}^{n} \frac{\Gamma(\beta^k_j + sB_i)}{\Gamma(b_i + B_i - sB_i)} \prod_{j=1}^{m} \frac{\Gamma(a_j - sA_j)}{\Gamma(a^k_j - A_j + sA_j)} \tag{3.9}$$

3.1.3 Reduction of (3.6) and (3.7) to Equations with A Common Kernel

The well known beta function can be expressed as

$$\int_o^x v^{c_m a_m - s - 1}\left(x^{c_m} - v^{c_m}\right)^{(\alpha_m - a_m - s - 1)} dv$$

$$= \frac{\Gamma(\alpha_m - a_m)\Gamma(a_m - sA_m)}{c_m \Gamma(\alpha_m - sA_m)} x^{c_m \alpha_m - c_m - s} \tag{3.10}$$

$c_m = \dfrac{1}{A_m}$, $\alpha_m > a_m$ and $c_m a_m = \dfrac{a_m}{A^m} > \sigma_o + it$ on the line $\sigma = \sigma_o$.

These conditions may no longer be necessary and the second may be relaxed. Replacing x by v in (3.6), multiplying by $_v c_m a_m - 1(_x c_m - _v c_m)$ $(\alpha^k{}_m - a^m - 1)$ and integrating (which is justified under the assumptions already made) through the integral sign with respect to v from 0 to x, $0 < x < 1$, we find that

$$\frac{1}{2\pi i}\int_L \chi^{(1)}_{m,n}(s) x^{-s} \sum_{h=1}^{n} a_{hk} F(1-s) ds$$

$$\frac{C_m x^{\left(c_m - \alpha^m_m c_m\right)\left(\frac{c_m a_m - 1}{v}\right)\left(\frac{c_m c_m}{x-v}\right)\alpha^k_m - a_m - 1}}{\Gamma(\alpha^k_m - a_m)} \phi_k(V) dv \tag{3.11}$$

where $\chi^{(1)}_{m,n}(s)$ is obtainable from $\chi^{(*)}_{m,n}(s)$ by replacing β^k_m by a_m only.

The operator of fractional integration denoted by R is used in the form

$$R[\lambda,\delta : m : w(x)] = \frac{m}{\Gamma(w)} x^{-\delta+m\lambda+m^{-1}}$$

$$= \int_0^x (x^m - v^m)^{\lambda-1} v^{\delta} w(v) dv \qquad (3.12)$$

The case m = 1 is due to Kober [48] and the result for more general case m > 0 has been given by Erdelyi [34], Fox [41] has shown that there is no essential difference between the two cases, since

$$R[y,\delta : m : w(x)] = R\left[y, m^{-1}(\delta+1)-1 : 1W(x)\right]. \qquad (3.13)$$

where $x^m = X$, $v^m = V$ and $w(x) = W(x)$.

If in addition, the operator R(x) exists provided $w(x)\varepsilon\ L_p(0,\infty)$, $\rho \geq 1$, $\gamma > 0$, $\delta > 1/q$. If w(x) can be differentiated sufficiently often, then the operator R exists for negative as well as positive values of γ.

For brevity, we write

$$R\left[(\alpha_1^k - a_i)(a_i A_i^{-1} - 1), A_i^{-1} : w(x)\right] = R_i[W(x)]. \qquad (3.14)$$

$$R\left[(b_j - \beta_j^k)(\beta_j^k - B_j) B_j^{-1} - 1, B_j^{-1} : w(x)\right] = R_j^*[w(x)]. \qquad (3.15)$$

From (3.14) it is seen that right hand side of (3.11) is equal to $R_m[\phi_k(x)]$ with $0 < x < 1$. On transforming the equation (3.6) step by step by means of the operators R_i and R_j^* for I = m, m-1.......2, 1 and j = n, n-1,

.....2, 1 we find that

$$\frac{1}{2\pi i}\int_L \chi^{(1)}_{m,n}(s)x^{-s}\sum_{h=1}^{n} a_{hk}F(1-s)ds$$

$$=R_1^*\left[R_2^*\ldots\ldots R_n^*,R_1\ldots\ldots R_m[\phi_k(x)]\ldots\ldots\right] \qquad 0<x<1.$$

We now proceed to transform the integral equations (3.7). Again from beta function formula, we have

$$\int_x^\infty v^{d_n d_n \beta_n^{k-s-1}}\left(v^{d_n}-x^{d_n}\right)\beta_n^k-b_n^{-1}dv=\frac{\Gamma\left(\beta_n^k-b_n\right)\Gamma\left(b_n+sB_n\right)}{d_n\Gamma\left(\beta_n^k+sB_n\right)}$$

where $d_n=B_n^{-1},\beta_n^k>b_n$ and $d_n\sigma_0+b_n>0$. In (3.7) let us replace x by v and multiply by $v^{dn-dn\beta_n k-s-1}\left(v_n^d-x_n^d\right)\beta_n^k-b_n^{-1}$ and integrate with respect to v from x to ∞ through the integral sign. Equation (3.7) takes the form

$$\frac{1}{2\pi i}\int_L \chi^{(2)}_{m,n}(s)x^{-s}\sum_{h=1}^{n} b_{hk}F_h(1-s)ds$$

$$=\frac{d_n x^{db}}{\Gamma\left(\beta_n^k-b\right)}\int_x^\infty vd_n-\beta_n^k d_n^{-1}(vd_n-xd_n)\beta_{nn}^{k-b-1}\phi_k(v)dv, \qquad (3.18)$$

where $x>1$, $d_n=\beta_n^{-1}$ and $\chi^{(2)}_{m,n}(s)$ is obtainable from $\chi^{**}_{m,n}(s)$ by the replacement of β_n^k by b_n. The second operator of fractional integration denoted by K required here is

$$K[\gamma,\delta:n:w(x)] = \frac{\eta}{\gamma^{1/2}},x^{\delta}\int_{x}^{\infty}(v^{n}-x^{n})^{\gamma-1}v^{-\delta-n\gamma+n-1}w(v)dv \qquad (3.19)$$

If w(x) $L_\rho(0,\infty)$, $\rho > 1$ and w(x) can be differentiated sufficiently often K exists, provided $n > 0$, $\delta > 1/\rho$, where γ can have any positive or negative values.

$$K[(\beta_j^k - b_i)b_i B_i^{-1}; B_i w(x)] = k_i[w(x)] \qquad (3.20)$$

and

$$K[(a_j - a_j^k)(\alpha_j^k A_j)A_j^{-1} : w(x)] = k_j^*[w(x)] \qquad (3.21)$$

from (3.20) it is evident that the right hand side of (3.18) is $K_n[\psi_k(x)]$, where $x > 1$.

The successive application of the operators, K_I for I = n, n-1,.......2, 1 and K_j^* of j = m, m-1,.....2, 1 to (3.7) yields the integral equation

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)x^{-s}\sum_{h=1}^{n} a_{hk}F_h(1-s)ds$$

$$= \sum_{h=1}^{n} c_{hk}K_1^*[K_2^*.....K_m^*K_1K_2......K_n[\psi_k(x)].......] \qquad (3.22)$$

where c_{hk} are the elements of the matrix $[a_{hk}]$ $[b_{hk}]^{-1}$

If we set

$$R_1^*[R_2^* \ldots . R_n^* R_1 R_2 \ldots .. R_m[\phi_k(x)] \ldots \ldots] \qquad 0 < x < 1 \qquad (3.23)$$

$$= \sum_{h=1}^{n} c_{hk} K_1^*[K_2^* \ldots . K_n^* K_1 K_2 \ldots .. K_m[\psi_k(x)] \ldots \ldots] \qquad x > 1 \qquad (3.22)$$

$k = 1,2,\ldots\ldots,n.$

Equations (3.16) and (3.22) can be put into the compact form as

$$\frac{1}{2\pi i}\int^{L} \chi_{m,n}(s) x^{-s} \sum_{h=1}^{n} a_{hk} F_h(1-s) ds$$

$$= \frac{1}{2\pi i}\int_{L} M[H_{2m,2n}(u)] x^{-s} \sum_{h=1}^{n} a_{hk} F_h(1-s) ds = g_k(x) \qquad (3.24)$$

where M denotes the Mellin transforms. Since $H_{2m,2n}^{n,m}(xu)$ is a symmetrical Fourier kernel, the formal solution can be readily obtained as

$$f_h(x) = \sum_{k=1}^{n} d_{hk} \int_0^{\infty} H_{2m,2n}^{n,m}\left[(xu)\Big|_{(b_n,B_n),(1-b_n-B_n,B_n)}^{(1-a_m,A_m),(a_m-A_m,A_m)}\right]$$

$$g_k(u) du = \sum_{k=1}^{n} d_{hk} \int_0^{1} H_{2m,2n}^{n,m}\left[(xu)\Big|_{(b_n,B_n),(1-b_n-B_n,B_n)}^{(1-a_m,A_m),(a_m-A_m,A_m)}\right] \qquad (3.25)$$

$$R_1^*[R_2^* \ldots . R_n^* R_1 R_2 \ldots .. R_m[\phi_k(x)] \ldots \ldots] \qquad (3.23)$$

$$\int_1^{\infty} H_{2m,2n}^{n,m}\left[(xu)\Big|_{(b_n,B_n),(1-b_n-B_n,B_n)}^{(1-a_m,A_m),(a_m-A_m,A_m)}\right]$$

$$\sum_{h=1}^{n} c_{hk} K_1^* \left[K_2^* \ldots K_m^* K_1 K_2 \ldots K_m [\psi_k(x)] \ldots\right] \tag{3.22}$$

where h = 1, 2,……,n.

3.1.4 Interesting Particular Cases

(1) If we set m = 0, n = 1, $B_1 = B_2 = 1$, $b_1 = \lambda^k + \gamma^k/2$, $\beta_1^k = \gamma_2^k - \lambda$ and use the identity []

$$G_{0,2}^{1,0}(x / a, b) = x^{a+b/2} j_{a-b}(2x)^{1/2} \tag{3.26}$$

We find the formal solution of the simultaneous dual integral equations

$$\int_0^{\infty} (ux)\lambda^k v^k \left[2(xu)^{1/2}\right] \sum_{h=1}^{n} a_{hk} f_h(u) = \phi_k(x), \qquad 0 < x < 1 \tag{3.27}$$

$$\int_0^{\infty} (ux)^{-\lambda k} v^k \left[2(xu)^{1/2}\right] \sum_{h=1}^{n} a_{hk} f_h(u) = \psi_k(x), \qquad x < 1 \tag{3.28}$$

k = 1,2,……,n;

is given by

$$f_h(x) = \sum_{h=1}^{n} d_{hk} \int_0^1 J v^k + 2\lambda^k \left[2(xu)^{1/2}\right] R\left[2\lambda^k, v^k / 2 - \lambda^k, 1\right] \tag{3.28a}$$

$$\phi_k^n(u) du + \int_{\infty}^{1} \sum J_{vk+2\lambda}^k \left[2(xu)^{1/2} \sum_{h=1}^{n} c_{hk} K\left[-2\lambda^k, \lambda^k + v^k / 2, 1,\right]\right]$$

$$\psi_k(u) du, \quad h = 1, 2, \ldots, n. \tag{3.29}$$

(ii) On the other hand if we put m = 1, n = 0, $A_1 = A_2 = 1$, $\alpha_1^k = 1 - \rho^k + \mu^k/2, a_1 = 1 + \rho^k + \mu^k/2$ and apply the transformation [2]

$$G_{p,q}^{m,n}\left[[x]_{b_p}^{a_p}\right] = G_{m,n}^{n,m}\left[\left[x^{-1}\right]_{1-a_p}^{1-b_q}\right], \tag{3.30}$$

$$G_{0,2}^{1,0}[[x/a,b]] = x^{(a+b)/2} J_{a-b}(2x)^{1/2}, \tag{3.31}$$

It is found that the formal solution of the simultaneous dual integral equations.

$$\int_0^\infty (ux)^{k-1} J_v k\left[2(xu)^{1/2}\right]\sum_{h=1}^{n} a_{hk} f_h(u)du = \phi_k(x), \qquad 0 < x < 1 \tag{3.32}$$

$$\int_0^\infty (ux)^{k-1} J_v k\left[2(xu)^{1/2}\right]\sum_{h=1}^{n} b_{hk} f_h(u)du = \psi_k(x), \qquad x > 1 \tag{3.32}$$

is given by

$$f_h(x) = \sum_{h=1}^{n} d_h k\left[\int_0^1 J_v k_{+2k}(xu)^{-1/2}\right]$$

$$R\left[-2_{q}^{k}{}_{,q}^{k} + \frac{\mu^k}{2}, 1, \phi_n(u)du + \int_1^\infty (xu)^{-1} J_v^k + 2\delta^k\right]$$

$$\left[2(xu)^{-1/2}\right]\sum_{h=1}^{n} c_{hk} K\left[2q^k, 2\mu^k - q^k, 1, \psi_k(u)du\right] \tag{3.34}$$

where h = 1, 2,………, n.

3.2 SIMULTANEOUS DUAL INTEGRAL EQUATIONS OF CONVOLUTION TYPE

This section deals with certain simultaneous dual integral equations of convolution type. The solution has been obtained by following the method of Tanno [117].

3.2.1 Introduction

Fox [41] and Saxena [91] have obtained the formal solution of certain dual integral equations involving H-function as kernel. Tanno [117] obtained the solution of more general class of dual integral equations. This class involves the case of Fox and Saxena as special cases.

Erdogan and Bahar [37] and Westmann [128] obtained the formal solution of simultaneous dual integral equations involving Bessel functions as kernel. Recently Dwivedi [21] has obtained the solution of certain simultaneous dual integral equations:

$$\int_0^\infty H^{m,n}_{p+n,q+n}\left[(xu)\Big|^{a_i^k, A_i}_{b_j^k, B_j}\right]\sum_{h=1}^{n} a_{hk} f_h(u)du = \phi_k(x), \tag{3.35}$$

$$\int_0^\infty H^{m,n}_{p+n,q+n}\left[(xu)\Big|^{c_i, A_i, c^k_{n+i}, A_{n+i}}_{d_j^k, B_j, d_{m+j}, B_{m+j}}\right]\sum_{h=1}^{n} b_{hk} f_h(u)du = \psi_k(x),$$

$$x > 1;\ k = 1, 2, 3, \ldots\ldots\ldots, n \tag{3.35}$$

and

$$\int_0^\infty H^{q,p+m}_{2p+m,2q+m}(xu)\sum_{h=1}^{n} a_{hk} f_h(u)du = \phi_k(x), \qquad 0 < x < 1, \qquad (3.37)$$

$$\int_0^\infty H^{q+n,p}_{2p+n,2q+n}(xu)\sum_{h=1}^{n} b_{hk} f_h(u)du = \psi_k(x), \qquad x > 1, \qquad (3.38)$$

$k = 1, 2, \ldots\ldots, n.$

where $H^{m,n}_{p,q}(x)$ is used to denote the function $H^{m,n}_{p,q}(Ha_iA_i)a_{hk}zb_{j,}B_j$ and b_{hk} are known constant $\phi_k(x)$ and $\psi_k(x)$ are given functions and $f_h(x)$ to be determined.

The H-functions of order n used above are of the form

$$H(x)\begin{pmatrix}\alpha_i, a_i\\ \beta_i, a_i\end{pmatrix}, n = H\left(x \,/\, \begin{matrix}\alpha_1 a_1, \alpha_2 a_2, \ldots\ldots\ldots\ldots \alpha_n, a_n\\ \beta_1 a_1, \beta_2 a_2, \ldots\ldots\ldots\ldots \beta_n, a_n\end{matrix}\right)$$

$$= \frac{1}{2\pi i} \int_{ci=1}^{n} \frac{\pi\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)} x^{-s} ds \qquad (3.39)$$

where $\alpha_1 > 0$, α_1, β_1 are all real, $i = 1, 2, \ldots\ldots, n$, the contour C along which the integral of (3.39) is taken as the straight line parallel to the imaginary axis in the complex s-plane and lie to the left to the line $\sigma = \sigma_0 > -\alpha / a$, $(s = \sigma + i\tau)$. The integral (3.39), taken along the line, coverages if

$$2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n} (\beta_i - \alpha_i)$$

and coverages absolutely if

$$2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n} (\beta_i - \alpha_i) - 1$$

Our object there is to obtain the solution of more general class of simultaneous dual integral equations of convolution type by following the method of Tanno [117].

From the point of view the simultaneous dual integral equations (3.35) (3.36) and (3.37), (3.38) can be reduced to the dual convolution transforms after exponential change of variables we consider the followings:

$$\int_{-\infty}^{\infty} G_k(x-t) \sum_{h=1}^{n} a_{hk} \phi_h(t) dt = f_k(x), \qquad x > \lambda \tag{3.40}$$

$$\int_{-\infty}^{\infty} H_k(x-t) \sum_{h=1}^{n} b_{hk} \psi_h(t) dt = g_k(x), \qquad x > \lambda \tag{3.41}$$

$$k = 1, 2, 3, \ldots\ldots, n.$$

where λ, a_{hk} and b_{hk} are constants and $G_k(t)$ and $H_k(t)$ are generated by certain meromorphic functions. The explicit forms of these functions are defined in the following section.

3.2.2 Definition of Kernel

Let us define admissible entire functions $E_i(s)$ $(i = 1, 2, 3, 4)$:

$$E_i(s) = \exp(\delta_i s) \prod_{j=1}^{\infty} \left(\frac{1-s}{a_{bj}} \right) \exp \frac{(s)}{a_{ij}}, \tag{3.42}$$

where δ_1, $a_{i,j}$ are real numbers such that $\sum_{k=1}^{\infty}\left\{\frac{1}{a_{ij}}\right\}^2 < \infty$ (i = 1, 2, 3, 4, j = 1, 2, ...) and $a_{i,j} < 0$, $a_{2,j} > 0$, $a_{3,j} < 0$, $a_{4,j} > 0$ (j = 1, 2,..........).

Let (a, b) be an interval included in the corresponding intervals I_i (I = 1, 2, 3, 4) which contain no zero of $E_i(s)$ except perhaps at an end point. Further, if

$$\int_{-\infty}^{\infty}\left|\frac{E_2^k(\sigma+i\tau)}{E_1(\sigma+i\tau)}\right| d\tau < \infty \text{ for every } \sigma \text{ in } a < \sigma < b,$$

$$\int_{-\infty}^{\infty}\left|\frac{E_4(\sigma+i\tau)}{E_3^k(\sigma+i\tau)}\right| d\tau < \infty \text{ for every } \sigma \text{ in } a < \sigma < b,$$

$$\lim_{\tau\to\infty}\frac{E_2^k(\sigma+i\tau)}{E_1(\sigma+i\tau)} dT = 0 \text{ uniformaly in } a < \sigma < b,$$

$$\lim_{\tau\to\infty}\frac{E_4(\sigma+i\tau)}{E_3(\sigma+i\tau)} = 0 \text{ uniformaly in } a < \sigma < b, K = 1, 2, 3, \ldots\ldots, n.$$

then there exist two function $sF_k(t)$ and $H_k(t)$ such that

$$\frac{E_2^k(s)}{E_1(s)} = \int_{-\infty}^{\infty} e^{-st} G_k(t) dt, \qquad a < \sigma < b, \tag{3.43}$$

$$\frac{E_4(s)}{E_3^k(s)} = \int_{-\infty}^{\infty} e^{-st} H_k(t) dt, \qquad a < \sigma < b, \tag{3.44}$$

K = 1, 2, 3,......, n.

where E_2^k and E_3^k are obtained from (3.42) replacing $a_{2}, {}_ja_{3}, {}_j$ by $a_2^k, {}_ja_3^k, {}_j$ respectively and the intervals converging absolutely. The classical inversion formula [] gives $F_k(t)$ and $H_k(t)$ respectively.

$$G_k(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{E_2^k(s)}{E_1(s)} e^{st} ds, \qquad -\infty < t < \infty,\ a < c < b \tag{3.45}$$

$$H_k(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{E_4(s)}{E_3^k(s)} e^{st} ds, \qquad -\infty < t < \infty,\ a < c < b \tag{3.46}$$

the integral converging absolutely.

Thus we may consider the simultaneous dual convolution transforms (3.40) and (3.41) of secion 3.2.1, where $f_k(x)$ and $g_k(x)$ are given and $\phi_h(x)$ are to be determined.

3.2.3 The Reduction of (3.40) and (3.41) to Transform with A Common Kernel

As usual, we reduce two given transforms with a common kernel. Then problem is reduced to inverting the sole convolution transform. To attain this object we suppose further that

$$\sum_{j=1}^{\infty}\left(\frac{1}{a_{1,j}} - \frac{1}{a_{3,j}}\right), \sum_{j=1}^{\infty}\left(\frac{1}{a_{2,j}} - \frac{1}{a_{4,j}}\right),$$

converge and that

$$= \delta_1 - \delta_3 + \sum_{j=1}^{\infty}\left(\frac{1}{a_{1,j}} - \frac{1}{a_{3,j}}\right) = \delta_2 - \delta_4 + \sum_{j=1}^{\infty}\left(\frac{1}{a_{2,j}} - \frac{1}{a_{4,j}}\right).$$

Now as in proceeding section, if

$$\int_{-\infty}^{\infty}\left|\frac{E_4(\sigma+i\tau)}{E_2^k(\sigma+i\tau)}\right| d\tau < \infty \text{ for every } \sigma \text{ in } a < \sigma < b,$$

$$\int_{-\infty}^{\infty}\left|\frac{E_3^k(\sigma+i\tau)}{E_1(\sigma+i\tau)}\right| d\tau < \infty \text{ for every } \sigma \text{ in } a < \sigma < b,$$

$$\lim_{|\tau|\to\infty}\frac{E_4(\sigma+i\tau)}{E_2^k(\sigma+i\tau)} = 0 \text{ uniformaly in } a < \sigma < b,$$

$$\lim_{T\to\infty}\frac{E_3^k(\sigma+i\tau)}{E_1(\sigma+i\tau)} = 0 \text{ uniformaly in } a < \sigma < b,$$

Then

$$G_k^*(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{E_4(s)}{E_2^k(s)}e^{st}ds, \qquad -\infty < t < \infty,\ a < c < b \tag{3.47}$$

$$H_k^*(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{E_3^k(s)}{E_1(s)}e^{st}dt, \qquad -\infty < t < \infty,\ a < c < b \tag{3.48}$$

are well defined and

$$\int_{-\infty}^{\infty}G_k^*(t)e^{-st}dt = \frac{E_4(s)}{E_2^k(s)}, \qquad a < \sigma < b, \tag{3.49}$$

$$\int_{-\infty}^{\infty} H_k^*(t)e^{-st}dt = \frac{E_3^k(s)}{E_1(s)}, \qquad a < \sigma < b, \tag{3.50}$$

the integrals converge absolutely.

We know that G_k^* and $H_k^*(t)$ are frequency functions and that

$$G_k^*(t)\begin{cases} o & t \geq \Omega \\ q_k(t)\exp\left(\alpha_2^{*k}t\right) + 0\left(\exp\left(\alpha_2^{*k} + \varepsilon\right)\right) & t \to -\infty \end{cases} \tag{3.51}$$

$$G_k^*(t)\begin{cases} P_k(t)\exp\left(\alpha_1^{*k}t\right) + 0\left(\exp\left(\alpha_1^{*} - \varepsilon\right)\right) & t \to +\infty \\ o & t \leq \Omega \end{cases} \tag{3.52}$$

for $\varepsilon > 0$, $k = 1, 2, \ldots, n$, where $p_k(t)$ are real polynomials and $\alpha_1^* = \max a_{1,j}$, α_2^{*k} $\min a_{2,j}^k (j = 1, 2, \ldots)$.

Since (a, b) is include in $I_1 \cap I_2 \cap I_3 \cap I_4$, it follows that the bilateral Laplace transforms of $G_k(t)$ and $G_k^*(t)$ and $H_k(t)$ and $H_k^*(t)$ have common region of absolute convergence and hence by the product theorem.

$$\frac{E_4(s)}{E_1(s)} = \int_{-\infty}^{\infty} e^{-st}G_{k*}G_k^*(t)dt = \int_{-\infty}^{\infty} e^{-st}H_{k*}H_k^*(t)dt \tag{3.53}$$

both integrals converging absolutely.

Using the uniqueness theorem of bilateral Laplace transforms, we obtain

$$G_{k*}G_k^*(x) = H_{k*}H_k^*, \qquad k = 1, 2, \ldots\ldots\ldots, n. \tag{3.54}$$

for all x. Hence we take the convolution of both sides of (3.40) and $G_k^*(x)$ and of (3.41) and $H_k^*(x)$ formaly and we have,

$$G_k^* f_k(x) = \int_{-\infty}^{\infty} G_k^*(x-u) f_k(u)\,du \tag{3.55}$$

$$= \int_{x-\Omega}^{\infty} G_k^*(x-u) f_k(u)\,du$$

$$= \int_{x-\Omega}^{\infty} G_k^*(x-u) \int_{-\infty}^{\infty} g_k(u-t) \sum_{h=1}^{n} a_{hk} \phi_h(t)\,dt$$

$$= \int_{-\infty}^{\infty} G_{k*} G_k^*(x-t) \sum_{h=1}^{n} a_{hk} \phi_h(t)\,dt$$

$x > \lambda + \Omega$, $k = 1, 2, \ldots\ldots, n$.

and

$$\sum_{h=1}^{n} c_{hk} \left[H_{k*}^* g_k(u)\right] = \sum_{h=1}^{n} c_{hk} \int_{-\infty}^{\infty} H_k^x(x-u) g_k(u)\,du \tag{3.56}$$

$$= \sum_{h=1}^{n} c_{hk} \int_{-\infty}^{x-\Omega} H_k^*(x-u) g_k(u)\,du$$

$$= \sum_{h=1}^{n} c_{hk} \int_{-\infty}^{x-\Omega} H_k^*(x-u) \int_{-\infty}^{\infty} H_k(u-t) \sum_{h=1}^{n} b_{hk} \phi_h(t)\,dt$$

$$= \int_{-\infty}^{\infty} \left[H_{k^-}^{x}, H_k^{x}(x-t)\sum_{h=1}^{n} a_{Rk}\ \phi_h(t)\,dt \right]$$

$x > \lambda + \Omega$, $k = 1, 2, 3, \ldots\ldots, n.$

where c_{hk} are the elements of the matrix $[a_{hk}][bhk]^{-1}$.

Therefore, if we set

$$h(x) = \begin{cases} \left[G_{k*}^{*} f_k(x)\right] & x > \lambda + \Omega \\ \sum_{h=1}^{n} c_{hk} \left[H_{k*}^{*} g_k(x)\right] & x < \lambda + \Omega \end{cases} \tag{3.57}$$

The dual convolution transforms (3.40) and (3.41) can be reduced to be transform with common kernel $G_{k*}\ G_{k*}$.

$$h(x) = \int_{-\infty}^{\infty} \left[G_{k*} G_k^{*}(x-t)\sum_{h=1}^{n} a_{hk}\phi_h(t)dt \right] \tag{3.58}$$

3.2.4 The Solution of (3.4) and (3.41)

Now if, $\phi_h(t)$ is continuous, $-\infty < t < \infty$ and when $(a,b)\varepsilon$

$$I_1 \cap I_2 \cap I_3 \cap I_4, \phi_h(t) = 0\left(e^{\alpha t}\right) \qquad t \to -\infty$$
$$= O\left(e\beta^k t\right) \qquad t \to \infty, a < \alpha < \beta^k < t$$

then on using Widders theorem [129] we can invert equation (3.57) to get $\phi_h(t)$, that is

$$G_k(t) = \frac{\mu}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{E_2^k(s)}{E_1(s)} e^{st} ds, \qquad -\infty < t < \infty,\ a < c < b$$

$$\phi_h(t) = \sum_{h=1}^{n} d_{hk} \frac{E_1(D)}{E_4(D)} \begin{cases} \left[G_{k*}^{*} f_k(x)\right] & x > \lambda + \Omega \\ \sum_{h=1}^{n} c_{hk} \left[H_{h*}^{*} g_k(x)\right] & x < \lambda + \Omega \end{cases}$$

where D stands for differentiation and the operator $E_1(D)/E_4(D)$ must be interpreted as usual meaning, and d_{hk} are the elements of matrix $[a_{hk}]^{-1}$.

3.2.5 Applications

To avoid unnecessary confusion we consider

$$\int_0^\infty H\left(ux\Big|_{\beta^{k,a}}^{\alpha,a};1\right) \sum_{h=1}^{n} a_{hk} f_h(u) du = g_x(x), \qquad 0 < x < 1 \qquad (3.59)$$

$$\int_0^\infty H\left(ux\Big|_{\mu,a}^{\lambda^k,a};1\right) \sum_{h=1}^{n} b_{hk} f_h(u) du = h_x(x), \qquad x > 1 \qquad (3.60)$$

$$k = 1, 2, \ldots\ldots, n.$$

In this case, the Mellin transform of $H\left(x\Big|_{\beta^{k,a}}^{\alpha,a};1\right)$ with variables $s(= \sigma + i\tau)$ coverges if $2\sigma a < (\beta^k - \alpha) - 1$ and its generating function is $\Gamma(\alpha + sa)/\Gamma(\beta^k - sa)$ and for the H-function of (3.60) same results with α replaced by λ^k and β^k replaced by μ hold.

After an exponential change of variables (3.59) and (3.60) become

$$\int_{-\infty}^{\infty} H\left(e^{-(x-t)} \Big|_{\beta^{k,a}}^{\alpha,a} ;1 \right) \sum_{h=1}^{n} a_{hk} f_h\left(e^t\right) e^t dt = g_k\left(e^{-x}\right), \qquad x > 0 \qquad (3.61)$$

$$\int_{0}^{\infty} H\left(e^{-(x-t)} \Big|_{\mu,a}^{\lambda^k,a} ;1 \right) \sum_{h=1}^{n} b_{hk} f_h\left(e^t\right) e^t dt = h_k\left(e^{-x}\right), \qquad x < 0 \qquad (3.62)$$

$$k = 1, 2, \ldots\ldots, n.$$

which are of the form (3.40) and (3.41) with $\phi_h(t) = f_h(e^t)\, e^t$.

From these facts, it is easily seen that entire function, defined in section 3.2.2 are

$$E_1(s) = \frac{1}{\Gamma(\alpha + sa)}, \quad E_2^k(s) = \frac{1}{\Gamma\left(\beta^k - sa\right)}$$

$$E_3^k(s) = \frac{1}{\Gamma\left(\lambda^k + sa\right)}, \quad E_k(s) = \frac{1}{\Gamma(\mu - sa)}$$

and that

$$a_{1,j} = -\frac{\alpha + j}{a}, \quad a_{2,j}^k = \frac{\beta^k + j}{a}, \quad a_{3,j}^k = \frac{\lambda^k + j}{a}, \quad a_{4,j} = \frac{\mu + j}{a},$$
$$k = 1,2,3,\ldots\ldots,n \text{ and } j = 0,1,2,3,\ldots\ldots$$

Here it is assumed that $\alpha > 0, \beta^k > 0, \lambda^k > \alpha + 1, \mu > \beta^k + 1$ and from familiar infinite expansions of Gamma function, it is clear that Ω defined in section 3.2.2 is zero.

After simple calculation, we have

$$G_k^*(t)\begin{cases}\dfrac{b}{\Gamma\left(\mu-\beta^k\right)}\left(1-e^{bt}\right)^{\mu-\beta k-1}e\left(b\beta^k t\right), & t<0\\ o & t>0\end{cases} \tag{3.63}$$

$$H_k^*(t)\begin{cases}o & t<0\\ \dfrac{b}{\Gamma\left(\lambda^k-\alpha\right)}\left(1-e^{-bt}\right)^{\lambda^k-\alpha-1}e-\left(b\alpha t\right), & t>0\end{cases} \tag{3.64}$$

where $b = 1/a$, $k = 1, 2, 3, \ldots\ldots, n$.

In order to reduce given simultaneous dual equations (3.59) and (3.60) to which have a common kernel, we can use two operators of fractional integration defined by τ and R, given by Fox [41] which after an exponential change of variables can be written as

$$\tau w\left(e^{-x}\right)=\frac{b}{\Gamma\left(\mu-\beta^k\right)}\int_x^{\infty}\left(1-e^{b(x-v)\mu}\right)\beta^{k-1}e^{b}\beta^{k(x-v)}w\left(e^{-v}\right)dv,\quad 0<x<1 \tag{3.65}$$

$$Rw\left(e^{-x}\right)=\frac{b}{\Gamma\left(\lambda^k-\beta\right)}\int_{-\infty}^{x}\left(1-e^{b(x-v)}\right)\lambda^{k-\alpha-1}e^{b\alpha(x-v)}w\left(e^{-v}\right)dv,\quad x<1 \tag{3.66}$$

Thus from (3.63) and (3.64), it is clear that

$$\tau w\left(e^{-x}\right)=\int_x^{\infty}G_k^*(x-v)w\left(e^{-v}\right)dv=G_{k*}^*e\left(e^{-x}\right) \tag{3.67}$$

Therefore from (3.67), we have

$$G^{*}_{k*}g_k\left(e^{-x}\right) = \tau g_k\left(e^{-x}\right) \tag{3.68}$$

Similarly from (3.64) and (3.66), we get

$$\sum_{h=1}^{n} c_{hk}\left(H^{*}_{k*}h_k\left(e^{-x}\right)\right) = \sum_{h=1}^{n} c_{hk} R h_k\left(e^{-x}\right).$$

Hence the solution of (3.61) and (3.62) is given by

$$\phi_h(t) = \sum_{h=1}^{n} d_{hk} \begin{matrix} E_1(D) \\ E_4(D) \end{matrix} \left| \begin{matrix} \left[\tau g_k\left(e^{-x}\right)\right], & x>\lambda+\Omega \\ \sum\limits_{h=1}^{n} c_{hk} R\left[h_k\left(e^{-x}\right)\right], & x<\lambda+\Omega \end{matrix} \right.$$

$k = 1, 2, \ldots\ldots. n.$

where d_{hk} are the elements of $[a_{hk}]^{-1}$.

Similarly, we can solve the equations (3.59) and (3.60) of order n by operating τ and R respectively.

CHAPTER – 4

TRIPLE INTEGRAL EQUATIONS

In this chapter we have considered the solution of certain triple integral equations with Legendre function as kernels by using the properties of generalized Legendre functions and inversion theorem for the generalized Mehler-Fock transform. This chapter contains brief introduction in section 4.1, some results in section 4.2, triple integral equations of the first and second kind in sections 4.3 and 4.4 respectively, some extensions of triple integral equations in section 4.5 and the more general triple integral equations in section 4.6 have been given.

4.1 INTRODUCTION

Triple integral equations arise in many problems of methematical physics [104]. Triple integral equations with Legendre functions as kernels were first considered by Srivastava [108]. He used a method similar to that of Cooke [13] to solve the each set of triple integral equations which has been reduced to a Fredholm integral equation of the second kind best treated by numerical method. Here we have considered more general equations than considered by Srivastava [108] or elsewhere. By using the properties of generalized Legendre functions and inversion theorem for the generalized Mohler-Fock transform, we have been able to reduce the two sets of triple integral equations to Fredholm integral equation of the second kind. Further extensions to include more general cases have also been discussed.

4.2 SOME RESULTS REQUIRED IN SEQUEL

It is convenient to list here some results for ready reference. The generalized Legendre function $P^{\mu}(z)$ is defined by [35].

$$P_{\nu}^{\mu}(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}{}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1}{2}-\frac{1}{2}z\right),1-z<2 \tag{4.2.1}$$

The following two integral representations for $P_{\nu}^{\mu}(z)$ are basic tools for our present investigations:

$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)=\left(\frac{\pi}{2}\right)^{-1/2}\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-1}\sin h^{\mu}\alpha$$
$$\int_0^{\alpha}\cos(t\tau)(\cos h\alpha-\cos ht)^{-\mu-\frac{1}{2}}dt \tag{4.2.2}$$

where $\operatorname{Re}(\mu)<\frac{1}{2}$ and

$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)=(2\pi)^{1/2}\left[\Gamma\left(\frac{1}{2}+\mu\right)\right]^{-1}\sin h^{-\mu}\alpha\operatorname{cosec}h(\pi\tau)$$
$$\left[\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\right]^{-1}\int_{\alpha}^{\alpha}\sin(t\tau)(\cos ht-\cos h\alpha)^{\mu-\frac{1}{2}}dt \tag{4.2.3}$$

where $-\frac{1}{2}<\operatorname{Re}(\mu)<\frac{1}{2}$.

Let $F_c(\tau)$ and $F_s(\tau)$ denote the Fourier cosine and sine transforms of $f(x)$. Then (4.2.2.) can be written as

$$F_c\left[U(\alpha-t)(\cos h\alpha-\cos ht)^{-\mu-\frac{1}{2}}\right]=\Gamma\left(\frac{1}{2}-\mu\right)\sin h^{-\mu}\alpha P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)\quad(4.2.4)$$

where U(x) is the Heavisided' unit function, so that by the inversion formula for the Fourier cosine e transform, we have

$$U(\alpha-t)(\cos h\alpha-\cos ht)^{-\mu-\frac{1}{2}}=(2/\pi)^{1/2}\Gamma\left(\frac{1}{2}-\mu\right)^{-1}\sin h^{-\mu}\alpha \int_0^{\infty}P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)\cos(t\tau)\,d\tau \quad (4.2.5)$$

Also (4.2.3) can be expressed as

$$F_s\left[U(\alpha-t)(\cos ht-\cos h\alpha)^{\mu-\frac{1}{2}}\right]=\pi^{-1}\Gamma\left(\frac{1}{2}+\mu\right) \Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\sin h^{\mu}\alpha\, P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha) \quad (4.2.6)$$

which on inversion gives

$$U(t-\alpha)(\cos ht-\cos h\alpha)^{\mu-\frac{1}{2}}=2^{1/2}\pi^{-3/2}\Gamma\left(\frac{1}{2}+\mu\right)\sin h^{\mu}\alpha \int_0^{\infty}\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\sin(\pi\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)\sin(t\tau)\,dt \quad (4.2.7)$$

If $\varphi(t)$ is monotonic strictly increasing and differentiable for $a<t<b$, and $\varphi'(t)\neq 0$ in this interval, then the solutions of the equations

$$\int_a^t \frac{f(x)dx}{[\varphi(t)-\varphi(x)]^{\alpha}}=g(t), \qquad a<t<b,\ 0<\alpha<1 \qquad (4.2.8)$$

and

$$\int_t^b \frac{f(x)dx}{[\varphi(x)-\varphi(t)]^{\alpha}} = g(t), \qquad a<t<b,\ 0<\alpha<1 \tag{4.2.9}$$

are given by.

$$f(x) = \pi^{-1}\sin(\pi\alpha)\frac{d}{dx}\int_a^x \frac{\varphi'(t)g(t)dt}{[\varphi(x)-\varphi(t)]^{1-\alpha}} \tag{4.2.10}$$

$$f(x) = -\pi^{-1}\sin(\pi\alpha)\frac{d}{dx}\int_x^b \frac{\varphi'(t)g(t)dt}{[\varphi(t)-\varphi(x)]^{1-\alpha}} \tag{4.2.11}$$

respectively. Therefore in view of (3.2.8) and (3.2.10), the integral representation (4.2.2) gives

$$\cos(t\tau) = (2\pi)^{-1/2}\Gamma\left(\frac{1}{2}-\mu\right)\cos(\mu\pi)\frac{d}{dt}\int_0^t \frac{\sin h^{1-\mu}\alpha\, P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)}{(\cos ht-\cos h\alpha)^{\frac{1}{2}-\mu}} \tag{4.2.12}$$

Also, in view of (4.2.9) and (4.2.11), from (4.2.3), we get

$$\cos(t\tau) = 2^{-1/2}\pi^{-3/2}\Gamma\left(\frac{1}{2}+\mu\right)\cos(\mu\pi)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)$$
$$+\sin h(\pi\tau)\int_t^{\infty} \frac{\sin h^{\mu+1}\alpha\, P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\alpha}{(\cos h\alpha-\cos ht)^{\frac{1}{2}+\mu}} \tag{4.2.13}$$

Another result that we shall need is the following inversion formula for the generalized Mehler-Fock transform due to Braaksma and Meulehbeld [Theorem 7] for conditions of velidity. If

$$\varphi(\cosh\alpha) = \int_0^\infty P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha) f(\tau) d\tau \tag{4.2.14}$$

then under certain conditions

$$f(\tau) = \pi^{-1}\tau \sinh(\pi\tau) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \Gamma\left(\frac{1}{2} - \mu - i\tau\right)$$
$$\int_0^\infty P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha) \varphi(\cos h\alpha) \sin h\alpha d\alpha \tag{4.2.15}$$

4.3 TRIPLE INTEGRAL EQUATIONS OF THE FIRST KIND

We with to solve here the triple integral equations of the first kind of the form

$$\int_0^\infty \varphi(\tau) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \sinh(\pi\tau)$$
$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha) d\tau = 0, \qquad 0 < \alpha < \alpha_1 \tag{4.3.1}$$

$$\int_0^\infty \varphi(\tau) P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha) d\tau = f(\alpha), \quad \alpha_1 < \alpha < \alpha_2 \tag{4.3.2}$$

$$\int_0^\infty \varphi(\tau) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \sinh(\pi\tau)$$
$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha) d\tau = 0, \qquad \alpha_2 < \alpha < \infty \tag{4.3.3}$$

where $f(\alpha)$ is known and $\varphi(\tau)$ is to be determined.

Now we assume that for $\alpha_1 < \alpha < \alpha_2$.

$$\int_0^{\infty}\varphi(\tau)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\tau\sinh(\pi\tau)$$

$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau=\chi(\alpha) \tag{4.3.4}$$

where $\chi(\alpha)$ is yet specified. On using the equations (4.2.14) and (4.2.15) in (4.3.4), we have

$$\varphi(\tau)=\pi^{-1}\int_{\alpha_1}^{\alpha_2}\chi(\alpha')\sinh\alpha' P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha')d\alpha' \tag{4.3.5}$$

Substituting $\varphi(\tau)$ in (4.3.2) and inverting the order of integration, we have

$$f(\alpha)=\int_{\alpha_1}^{\alpha_2}\chi(\alpha')L(\alpha,\alpha')\sinh\alpha' d\alpha' \tag{4.3.6}$$

where

$$L(\alpha,\alpha')=\frac{1}{\pi}\int_0^{\infty}P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha')d\tau \tag{4.3.7}$$

$$L(\alpha,\alpha')=\frac{1}{\pi}\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-2}\sin h^{\mu}\alpha\sin h^{\mu}\alpha'$$

$$\int_0^{\min(\alpha,\alpha')}\frac{dt}{(\cos h\alpha-\cosh t)^{\mu+1/2}(\cos h\alpha'-\cosh t)^{\mu+1/2}} \tag{4.3.8}$$

Substituting the value of $L(\alpha,\alpha')$ is equation (3.3.6), we get

$$f(\alpha)=\frac{\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-2}}{\pi}\int_{\alpha_1}^{\alpha_2}\chi(\alpha')\sin h^{1+\mu}\alpha'\sin h^{\mu}\alpha\,d\alpha' \tag{4.3.9}$$

$$\int_0^{\min(\alpha,\alpha')}\frac{dt}{(\cos h\alpha-\cosh t)^{\mu+1/2}(\cos h\alpha'-\cosh t)^{\mu+1/2}}$$

On changing the order of integration, we note that

$$\int_{\alpha_1}^{\alpha_2}d(\alpha')\int_0^{\min(\alpha,\alpha')}dt=\int_{\alpha_1}^{\alpha}d\alpha'\int_0^{\alpha'}dt+\int_{\alpha}^{\alpha_2}d\alpha'\int_0^{\alpha}dt=\int_{\alpha_1}^{\alpha}dt\int_t^{\alpha_2}d\alpha'+\int_0^{\alpha_1}dt\int_0^{\alpha}d\alpha'$$

On inverting the order of integration (the integral being understood). Hence we have (on replacing t by y in last integral).

$$\int_{\alpha_1}^{\alpha_2}\frac{dt}{(\cos h\alpha-\cosh t)^{\mu+1/2}}\int_t^{\alpha_2}\frac{\chi(\alpha')\sin h^{1+\mu}\alpha'\,d\alpha'}{(\cos h\alpha'-\cosh t)^{\mu+1/2}}$$

$$=\frac{\pi\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-2}}{\sin h^{\mu}\alpha}f(x)-\int_0^{\alpha_1}\frac{dy}{(\cos h\alpha-\cosh y)^{\mu+1/2}} \tag{4.3.10}$$

$$\int_{\alpha_1}^{\alpha_2}\frac{\chi(\alpha')\sin h^{1+\mu}\alpha'\,d\alpha'}{(\cos h\alpha'-\cosh t)^{\mu+1/2}}$$

If we now write that

$$G(t)=\int_t^{\alpha_2}\frac{\chi(\alpha')\sin h^{1+\mu}\alpha'\,d\alpha'}{(\cos h\alpha'-\cosh t)^{\mu+1/2}} \tag{4.3.11}$$

and hence equation (4.2.11) gives

$$\chi(\alpha')\sin h^{1+\mu}\alpha' = -\frac{1}{\pi}\sin\left[\mu+1/2\right]\frac{d}{d\alpha'}\int_{\alpha'}^{\alpha_2}\frac{\sinh t\, G(t)\,dt}{(\cosh t - \cosh\alpha')^{\frac{1}{2}-\mu}} \tag{4.3.12}$$

and by equation (4.3.10), we have

$$G(t) = \frac{1}{\pi}\sin\left[\pi\left(\mu+\tfrac{1}{2}\right)\right]\frac{d}{dt}\int_{\alpha_1}^{t}\frac{\sin h\alpha\, d\alpha}{(\cos ht - \cos h\alpha)^{\frac{1}{2}-\mu}}$$

$$\left[\frac{\pi\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^2 f(x)}{\sinh^{\mu}\alpha} - \int_0^{\alpha_1}\frac{dy}{(\cos h\alpha - \cosh y)^{\mu+1/2}}\right. \tag{4.3.13}$$

$$\left.\int_{\alpha_1}^{\alpha_2}\frac{\chi(\alpha')\sin h^{1+\mu}\alpha'\, d\alpha'}{(\cos h\alpha' - \cosh y)^{y+1/2}}\right]$$

On simplifying the last integral final result is

$$G(t) = \int_{\alpha_1}^{\alpha_2} G(s)K(s,t)\,ds = F(t) \tag{4.3.14}$$

where K(s,t) and F(t) is given by

$$K(s,t) = \frac{1}{\pi}\frac{\sin ht\, \sinh s\, \sin\left[\pi\left(\frac{1}{2}+\mu\right)\right]}{(\cos ht - \cosh\alpha_1)^{\frac{1}{2}-\mu}}\int_0^{\alpha_1}\frac{(\cos h\alpha_1 - \cosh y)^{\frac{1}{2}-\mu}\,dy}{(\cos ht - \cosh y)(\cos hs - \cosh y)} \tag{4.3.15}$$

and

$$F(t) = \left[\Gamma\left(\tfrac{1}{2}-\mu\right)\right]^2\sin\left[\pi\left(\mu+\frac{1}{2}\right)\right]\frac{d}{dt}\int_{\alpha_1}^{t}\frac{\sin h^{1-\mu}\alpha\, f(\alpha)\,d\alpha}{(\cos ht - \cosh\alpha)^{\frac{1}{2}-\mu}} \tag{4.3.16}$$

The equation (4.3.14) is a Fredholm integral equation of the second kind. The solution of the equation (4.3.1) to (4.3.3) follows from the

equations (4.3.5), (4.3.12) and (4.3.14).

4.4 TRIPLE INTEGRAL EQUATIONS OF THE SECOND KIND

Now we consider triple integral equations of the second kind of the form

$$\int_0^\infty \tau\varphi(\tau) P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = 0, \qquad 0<\alpha<\alpha_1 \tag{4.4.1}$$

$$\begin{aligned} &\int_0^\infty \varphi(\tau)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\sinh(\pi\tau) \\ &P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = f(x), \qquad \alpha_1<\alpha<\alpha_2 \end{aligned} \tag{4.4.2}$$

$$\int_0^\infty \tau\varphi(\tau) P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = 0, \qquad \alpha_2<\alpha<\infty \tag{4.4.3}$$

where $f(\alpha)$ is known and $\varphi(\tau)$ is to be determined. Let us assume that for $\alpha_1 < \alpha < \alpha_2$.

$$\int_0^\infty \tau\varphi(\tau) P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = g(\alpha) \tag{4.4.4}$$

where $g(\alpha)$ is yet to unspecified function. On using the equations (4.2.14) and (4.2.15) in (4.4.4), we have

$$\varphi(\tau) = \frac{1}{\pi}\Gamma\left(\frac{1}{2} - \mu + i\tau\right)\Gamma\left(\frac{1}{2} - \mu - i\tau\right)\sinh(\pi\tau)$$
$$\int_{\alpha_1}^{\alpha_2} P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha')g(\alpha')\sinh\alpha' d\alpha' \qquad (4.4.5)$$

Substituting the value of $\varphi(\tau)$ in (4.4.2) and inverting the order of integration, we have

$$f(\alpha) = \int_{\alpha_1}^{\alpha_2} g(\alpha')L_1(\alpha,\alpha')\sinh\alpha' d\alpha' \qquad (4.4.6)$$

where

$$L(\mu,\mu') = \frac{1}{\pi}\int_0^{\infty}\left[\Gamma\left(\frac{1}{2} - \mu + i\tau\right)\Gamma\left(\frac{1}{2} - \mu - i\tau\right)\right]^2 \sinh^2(\pi\tau)$$
$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha')d\tau \qquad (4.4.7)$$

Now using the equations (4.2.3) and (4.2.6), we have

$$L(\alpha,\alpha') = \frac{\pi\left[\Gamma\left(\frac{1}{2} - \mu\right)\right]^{-2}}{\sin h^{\mu}\alpha \sin h^{\mu}\alpha'}\int_{\max(\alpha,\alpha')}^{\infty}\frac{dt}{(\cos ht - \cosh\alpha')^{\frac{1}{2}-\mu}} \qquad (4.4.8)$$
$$\frac{1}{(\cos ht - \cosh\alpha)^{\frac{1}{2}-\mu}}$$

On using the value of $L_1(\alpha,\alpha')$, we get

$$f(\alpha)=\pi\int_{\alpha_1}^{\alpha_2}\frac{g(\alpha')}{\sin h^{\mu}\alpha}\frac{\sin h\alpha'\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-2}d\alpha'}{\sin h^{\mu}\alpha'} \tag{4.4.9}$$

$$\int_{\max(\alpha,\alpha')}^{\infty}\frac{dt}{(\cos ht-\cosh\alpha)^{\frac{1}{2}-\mu}(\cos ht-\cosh\alpha)^{\frac{1}{2}-\mu}}$$

On changing the order of integration, we note that

$$\int_{\alpha_1}^{\alpha_2}d\alpha'\int_{\max(\alpha,\alpha')}^{\infty}dt=\int_{\alpha_1}^{\alpha}d\alpha'\int_{\alpha}^{\infty}dt+\int_{\alpha}^{\alpha_2}d\alpha'\int_{\alpha'}^{\infty}dt=\int_{\alpha_2}^{\alpha_2}dt\int_{\alpha_1}^{t}d\alpha'+\int_{\alpha_2}^{\infty}dt\int_{\alpha_1}^{\alpha_2}d\alpha'$$

On inverting the order of integration.

Hence we have (on replacing t by y) in last integral.

$$\int_{\alpha_1}^{\alpha_2}\frac{dt}{(\cos ht-\cosh\alpha)^{\frac{1}{2}-\mu}}\int_{\alpha_1}^{t}\frac{g(\alpha')\sin h^{1-\mu}\alpha'\,d\alpha'}{(\cosh t-\cos h\alpha')^{\frac{1}{2}-\mu}}$$

$$=\frac{\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-2}}{\pi}\sinh^{\mu}\alpha\, f(x)-\int_{\alpha_2}^{\infty}\frac{dy}{(\cosh y-\cos h\alpha)^{\frac{1}{2}-\mu}} \tag{4.4.10}$$

$$\int_{\alpha_1}^{\alpha_2}\frac{g(\alpha')\sin h^{1-\mu}\alpha'\,d\alpha'}{(\cosh y-\cos h\alpha')^{\frac{1}{2}-\mu}}$$

If we now write that

$$G(t)=\int_{\alpha_1}^{t}\frac{g(\alpha')\sin h^{1-\mu}\alpha'\,d\alpha'}{(\cosh t-\cos h\alpha')^{\frac{1}{2}-\mu}} \tag{4.4.11}$$

and hence from (Abel integral equation) (4.2.11), we get

$$g(\alpha')\sin h^{1-\mu}\alpha' = \frac{1}{\pi}\sin\left[\pi\left(\frac{1}{2}-\mu\right)\right]\frac{d}{dt}\int_{\alpha_1}^{\alpha'}\frac{\sinh t\; G(t)\,dt}{(\cosh\alpha' - \cos ht)^{\frac{1}{2}+\mu}} \tag{4.4.12}$$

Now, by equation (4.4.10), we have

$$G(t) = -\frac{1}{\pi}\sin\left[\pi\left(\tfrac{1}{2}-\mu\right)\right]\frac{d}{dt}\int_{t}^{\alpha_2}\frac{\sin h\alpha d\alpha}{(\cos h\alpha - \cos ht)^{\frac{1}{2}-\mu}}$$

$$\left[\frac{\left[\left(\tfrac{1}{2}+\mu\right)\right]^2}{\pi}\sinh^{\mu}\alpha\, f(x) - \int_{\alpha_2}^{\infty}\frac{dy}{(\cosh y - \cos h\alpha)^{1/2-\mu}} \right. \tag{4.4.13}$$

$$\left.\int_{\alpha_1}^{\alpha_2}\frac{g(\alpha')\sin h^{1-\mu}\alpha' d\alpha'}{(\cosh y - \cos h\alpha')^{1/2-\mu}}\right]$$

On simplifying the last term, the final result is

$$G(t) + \int_{\alpha_1}^{\alpha_2} G(s) K_1(s,t)\,ds = F(t) \tag{4.4.14}$$

where $K_1(s,t)$ and $F(t)$ is given by

$$K_1(s,t) = \frac{\sin ht \sinh s \sin\left[\pi\left(\tfrac{1}{2}-\mu\right)\right](\cos h\alpha_2 - \cosh t)^{\frac{1}{2}+\mu}}{\pi(\cosh\alpha_2 - \cos hs)^{\frac{1}{2}+\mu}}$$

$$\int_{\alpha_2}^{\infty}\frac{dy}{(\cosh y - \cos ht)(\cosh y - \cos hs)} \tag{4.4.15}$$

and

$$F(t) = \frac{\left[\Gamma\left(\tfrac{1}{2}+\mu\right)\right]^2}{\pi^2}\frac{d}{dt}\int_{t}^{\alpha_2}\frac{\sin h^{1+\mu}\alpha\, f(\alpha)\,d\alpha}{(\cosh\alpha - \cos ht)^{\frac{1}{2}-\mu}} \tag{4.4.16}$$

The equation (4.4.14) is a Fredholm integral equation. Therefore, the solution of the equation (4.4.1) to (4.4.3) follows from the equations (4.4.5), (4.4.12) and (4.4.14).

4.5 EXTENSIONS

If the right hand sides of equations (4.3.1) and (4.3.3) are $h_1(\alpha)$ and $h_2(\alpha)$ respectively instead of zero, we may proceed as below.

First we determine a function $A(\tau)$ which is such that

$$\int_0^\infty A(\tau)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\tau\sinh(\pi\tau)$$
$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = h_1(\alpha), \qquad 0<\alpha<\alpha_1 \tag{4.5.1}$$

$$\int_0^\infty A(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = (\alpha), \qquad \alpha_1<\alpha<\alpha_2 \tag{4.5.2}$$

$$\int_0^\infty A(\tau)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\tau\sinh(\pi\tau)$$
$$P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)dt = h_2(\alpha), \qquad \alpha_2<\alpha<\infty \tag{4.5.3}$$

where $l(\alpha)$ is any convenient function of α. $A(\tau)$ can be determined by inversion from (4.2.15) and so we can find a function $m(\alpha)$ which is such that

$$\int_0^\infty A(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)d\tau = m(\alpha), \text{ for } \alpha_1<\alpha<\alpha_2 \tag{4.5.4}$$

Now write $B(\tau) = \varphi(\tau) - A(\tau)$ and subtract equations (4.5.1), (4.5.4) and (4.5.3) from equations (4.3.1), (4.3.2) and (4.3.3) in their now form and we have

$$\int_0^\infty B(\tau)\Gamma\left(\frac{1}{2}-\mu+ir\right)\Gamma\left(\frac{1}{2}-\mu-ir\right)\tau\sinh(\pi\tau)\, P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)d\tau = 0 \qquad 0<\alpha<\alpha_1 \tag{4.5.5}$$

$$\int_0^\infty B(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)d\tau = f(\alpha) - m(\alpha), \qquad \alpha_1 < \alpha < \alpha_2 \tag{4.5.6}$$

$$\int_0^\infty B(\tau)\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\Gamma\left(\frac{1}{2}-\mu-i\tau\right)\tau\sinh(\pi\tau)\, P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)dt = 0, \qquad 0<\alpha<\infty \tag{4.5.7}$$

which are of the form (4.3.1), (4.3.2) and (4.3.3) can thus be found and so $\varphi(\tau)$ is found.

4.6 THE MORE GENERAL TRIPLE INTEGRAL EQUATIONS

The equations considered in section 4.3 can be further generalized if there is an extra factor $[1 + H(\tau)]$ in equation (4.3.2) in its now form, where $H(\tau)$ is a known function. Thus case may be treated in the same case as done by Cooke [21] but it loads to multiple integrals, which are not specified. Then the final result [5.3.14] will be a Fredholm integral equation of the second kind with an extra term for the new form of the second kind with an extra terms for the new form of the equation. The extra term will be

$$-\int_{\alpha_1}^{\alpha_2} G(s)ds \int_0^\infty H(\tau)I(y,\tau)I(y,\tau)d\tau \tag{4.6.1}$$

where

$$I(t,\tau) = \Gamma\left(\frac{1}{2}-\mu\right)\sin\left[\pi+\frac{1}{2}\right]\frac{d}{dy}\int_{\alpha_1}^{y}\frac{\sin^{1-\mu}\alpha\, P^{\mu}_{-\frac{1}{2}+i\tau}(\cos h\alpha)dt}{(\cos hy-\cos h\alpha)^{\frac{1}{2}-\mu}} \tag{4.6.2}$$

CHAPTER –5

TRIPLE SERIES EQUATIONS INVOLVING ASSOCIATED LEGENDRE FUNCTION

5. INTRODUCTION

Triple series equations are useful in finding the solutions of three part mixed boundary value problems of electronics, elasticity and other field of mathematical physics [104] Collins [11] solved the triple series equations for the first time and since then many authors like Lowndes [55], Dwivedi and Gupta [27], Parihar [78], Melrose and Tweed [62] obtained the solutions of the triple series equations involving types of special functions. In the next section we find solution of some triple series equations involving associated Legendre function.

5.1 THE EQUATIONS

Here we are concerned with triple series equations of the form:

(i) ***Triple Series Equations of the First Kind***

$$\sum_{n=0}^{\infty}(2n+2m+1)A_n T_{m+n}^{-m}(\cos\theta)=f_1(\theta), \qquad 0\le\theta<\alpha \qquad (5.5.1)$$

$$\sum_{n=0}^{\infty} A_n T_{m+n}^{-m}(\cos\theta)=f_2(\theta), \qquad \alpha<\theta<\beta \qquad (5.5.2)$$

$$\sum_{n=0}^{\infty}(2n+2m+1)A_n T_{m+n}^{-m}(\mathrm{Cos}\theta)=f_3(\theta), \qquad \beta<\theta<\pi \qquad (5.5.3)$$

(ii) ***Triple Series Equations of the Second Kind***

$$\sum_{n=0}^{\infty}B_n T_{m+n}^{-m}(\mathrm{Cos}\theta)=g_1(\theta), \qquad 0\le\theta<\alpha \qquad (5.5.4)$$

$$\sum_{n=0}^{\infty}(2n+2m+1)B_n T_{m+n}^{-m}(\mathrm{Cos}\theta)=g_2(\theta), \qquad \alpha<\theta<\beta \qquad (5.5.5)$$

$$\sum_{n=0}^{\infty}B_n T_{m+n}^{-m}(\mathrm{Cos}\theta)=g_3(\theta), \qquad \beta<\theta<\pi \qquad (5.5.6)$$

In above equations $f_i(\theta)$ and $g_i(\theta)$, $(i = 1,2,3)$ are the prescribed functions of the variable θ and the equations (5.5.1) to (5.5.6) are to be solved for the unknown coefficients A_n and B_n. T_{m+n}^{-m} (Cos θ) is the associated legendre function of degree n+m and order −m of the first kind,

5.2 PRELIMINARY RESULTS

(i) ***Inversion Theorem for Associated Legendre Polynomials. If We Assume That The Expansion***

$$f(\theta)=\sum_{n=0}^{\infty}(2n+2m+1)C_n T_{m+n}^{-m}(\mathrm{Cos}\theta) \qquad (5.2.1)$$

is valid for $0 \le \theta \le \pi$ and that it can be integrated term by term, the coefficient C_n are given by

$$C_n = \frac{1}{2}(-)^m \int_0^{\pi} f(x) T_{m+n}^{m}(\mathrm{Cos}x)\mathrm{Sin}x\, dx \tag{5.5.2}$$

(ii) ***The Series***

$$S_m(\theta, x) = \sum_{n=0}^{\infty} T_{m+n}^{-m}(\mathrm{Cos}\theta) T_{m+n}^{m}(\mathrm{Cos}x) \tag{5.5.3}$$

$$= \frac{(-)^m}{2\pi} \sum_{n=0}^{\infty} \int_0^{2\pi} P_n(\mathrm{Cos}r)\mathrm{Cos}m\psi d\psi \tag{5.5.4}$$

$$= \frac{2(-)^m}{\pi(s_1 s_2)^m} \int_0^{\min(s_1,s_2)} \frac{s^{2m} ds}{\left[\left(s_1{}^2 - s^2\right)\left(s_2{}^2 - s^2\right)\right]^{1/2}} \tag{5.5.5}$$

Where $s_1 = 2\mathrm{Sin}\frac{\theta}{2}\mathrm{Cos}\frac{\theta}{2}x$, $\quad s_2 = 2\mathrm{Sin}\frac{1}{2}x\mathrm{Cos}\frac{1}{2}\theta$

And both s_1 and s_2 are positive for all θ and x since $0 < \theta$, $x < \pi$.

and $\quad S = 2\mathrm{Cos}\frac{\theta}{2}\mathrm{Cos}\frac{x}{2}\tan\frac{4}{2}$

(iii) ***Finally, We Require An Integral Representation of the Associated Legendre Polynomial*** T_{m+n}^{-m} ***(Cos*** θ***), to Obtain Which We Make Use of the Following Result:***

If, $$f(\theta) = \int_{\alpha}^{\theta} \frac{g(u)}{(\text{Cos}u - \text{Cos}\theta)^{1/2}} du, \qquad \alpha < \theta < \beta \qquad (5.5.6)$$

and $$f'(\theta) = \int_{\theta}^{\beta} \frac{g'(u)}{(\text{Cos}\theta - \text{Cos}u)^{1/2}} du, \qquad \alpha < \theta < \beta \qquad (5.5.7)$$

then $$g(u) = \frac{1}{\pi} \frac{d}{du} \int_{\alpha}^{u} \frac{f(\theta)\text{Sin}\theta}{(\text{Cos}\theta - \text{Cos}u)^{1/2}} d\theta, \qquad \alpha < u < \beta \qquad (5.5.8)$$

and $$g'(u) = \frac{1}{\pi} \frac{d}{du} \int_{u}^{\beta} \frac{f'(\theta)\text{Sin}\theta}{(\text{Cos}u - \text{Cos}\theta)^{1/2}} d\theta, \qquad \alpha < u < \beta \qquad (5.5.9)$$

THE SOLUTION

5.3 EQUATIONS OF THE FIRST KIND

In order to solve the triple series equations of the first kind, we set

$$\sum_{n=0}^{\infty} (2n + 2m + 1) A_n T_{m+n}^{-m} (\text{Cos}\theta) = h(\theta), \qquad \alpha < \theta < \beta \qquad (5.3.1)$$

Applying the inversion theorem (5.2.2) in equations (5.1.1), (5.1.3) and (5.3.1), we get

$$A_n = \frac{1}{2}(-)^m \left[\int_0^\alpha f_1(x) + \int_\alpha^\beta h(x) + \int_\beta^\pi f_3(x) \right] T_{m+n}^m (\operatorname{Cos}x) \operatorname{Sin}x \, dx \qquad (5.3.2)$$

Substituting the expression (5.3.2) for A_n in equation (5.1.2), we get

$$\sum_{n=0}^{\infty} T_{m=n}^{-m}(\operatorname{Cos}\theta).\frac{1}{2}(-)^m \left[\int_0^\alpha f_1(x) + \int_\alpha^\beta h(x) + \int_\beta^\pi f_3(x) \right],$$

$$\times T_{m+n}^m (\operatorname{Cos}x) \operatorname{Sin}x dx = f_2(\theta), \qquad \alpha < \theta < \beta \qquad (5.3.3)$$

Interchanging the order of summation and integration, we get

$$\frac{1}{2}(-)^m \left[\int_0^\alpha f_1(x) + \int_\alpha^\beta h(x) + \int_\beta^\pi f_3(x) \right]$$

$$\sum_{n=0}^{\infty} T_{m+n}^{-m}(\operatorname{Cos}\theta) T_{m+n}^m (\operatorname{Cos}x) \times \operatorname{Sin}x dx = f_2(\theta), \qquad \alpha < \theta < \beta$$

(5.3.4)

Using the expression given by (5.2.3) in (5.3.4), we obtain

$$\frac{1}{2}(-)m \int_\alpha^\beta h(x) S_m(\theta, x) \operatorname{Sin}x dx = F(\theta), \qquad \alpha < \theta < \beta \qquad (5.3.5)$$

where

$$F(\theta) = f_2(\theta) - \frac{1}{2}(-)^m \left[\int_0^\alpha f_1(x) S_m(\theta, x) \operatorname{Sin}x dx + \int_\beta^\pi f_3(x) S_m(\theta, x) \operatorname{Sin}x dx \right] \quad (5.3.6)$$

Now using the summation result in terms of integral (5.2.5) in the equation (5.3.5), we get

$$\frac{2.\frac{1}{2}(-)^{2m}}{\pi(s_1 s_2)^m}\left[\int_\alpha^\beta h(x)\int_0^{\min(s_1,s_2)}\frac{s^{2m}ds}{\left[\left(s_1^2-s^2\right)\left(s_2^2-s^2\right)\right]^{1/2}}\text{Sinxdx}\right]$$

$$= F(\theta), \qquad \alpha < \theta < \beta \qquad (5.3.7)$$

After breaking into parts, and noting that $s_1 > s_2$ when $\theta < x$, and $s_1 > s_2$ when $\theta > x$, we have

$$\frac{\sin^{-m}\theta}{\pi}\left[\int_\alpha^\theta h(x)\sin^{1-m}x\int_0^{s_2}\frac{s^{2m}ds}{\left[\left(s_1^2-s^2\right)\left(s_2^2-s^2\right)\right]^{1/2}}dx\right.$$

$$\left.+\int_\theta^\beta h(x)\sin^{1-m}x\int_0^{s_1}\frac{s^{2m}ds}{\left[\left(s_1^2-s^2\right)\left(s_2^2-s^2\right)\right]^{1/2}}dx\right] = F(\theta), \alpha < \theta < \beta \qquad (5.3.8)$$

Making change of variables by putting the values of s, s_1 and s_2, we get

$$\frac{\sin^{-m}\theta}{\pi}\left[\int_\alpha^\theta h(x)\sin^{1-m}x dx.\frac{1}{2}\int_0^x\frac{\left(2\cos\frac{1}{2}\theta.\cos\frac{1}{2}x.\tan\frac{1}{2}u\right)^{2m}}{[(\cos u-\cos\theta)(\cos u-\cos x)]^{1/2}}du\right.$$

$$\left.+\int_\theta^\beta h(x)\sin^{1-m}x dx.\frac{1}{2}\int_0^\theta\frac{\left(2\cos\frac{1}{2}\theta.\cos\frac{1}{2}x.\tan\frac{1}{2}u\right)^{2m}}{[(\cos u-\cos\theta)(\cos u-\cos x)]^{1/2}}du\right]$$

$$= F(\theta), \qquad \alpha < \theta < \beta \qquad (5.3.9)$$

Now inverting the order of integration, we get

$$\frac{2^{-m}\operatorname{Sin}^{-m}\frac{\theta}{2}.\operatorname{Cos}^{-m}\frac{\theta}{2}.2^{2m}\operatorname{Cos}^{2m}\frac{\theta}{2}}{2\pi}\left[\int_0^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du\right.$$

$$\times\int_{\alpha}^{\theta}\frac{2^{-m}h(x)\operatorname{Cot}^{+m}\frac{x}{2}.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$\times\int_{u}^{\theta}\frac{2^{-m}h(x)\operatorname{Cot}^{m}\frac{x}{2}.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_{0}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$\left.\times\int_{\theta}^{\beta}\frac{2^{-m}h(x)\operatorname{Cot}^{m}\frac{x}{2}.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx\right]=F(\theta),\qquad \alpha<\theta<\beta \tag{5.3.10}$$

Breaking the last term in to part, on the left hand side, we get

$$\frac{2^{2m}\tan^{-m}\frac{1}{2}\theta\operatorname{Cos}^{2m}\frac{1}{2}\theta}{2\pi}\left\{\int_0^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du\right.$$

$$x\int_{\alpha}^{\theta}\frac{h(x)\operatorname{Cot}^{m}\frac{x}{2}.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$x\int_{u}^{\theta}\frac{h(x)\operatorname{Cot}^{m}\frac{x}{2}.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_{0}^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$x\int_{\theta}^{\beta}\frac{h(x)\operatorname{Cot}^{m}\frac{x}{2}.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

Interchanging the order of summation and integration, we get

$$\frac{1}{2}(-)^m\left[\int_0^\alpha k_1(x)+\int_\alpha^\beta g_2(x)+\int_\beta^\pi k_2(x)\right]\sum_{n=0}^{\infty} T_{m+n}^{-m}(\mathrm{Cos}\theta)$$

$$T_{m+n}^{m}(\mathrm{Cos}x)\times \mathrm{Sin}x dx = \begin{cases} g_1(\theta), & 0\le\theta<\alpha \quad (5.4.6) \\ g_3(\theta), & \beta<\theta<\pi \quad (5.4.7) \end{cases}$$

Using the result (5.2.3) in equations (5.4.6) and (5.4.7), we have

$$\frac{1}{2}(-)^m\left\{\int_0^\alpha k_1(x)+\int_\beta^\pi k_2(x)\right\}S_m(\theta,x)\mathrm{Sin}\,x dx = \begin{cases} M(\theta), & 0\le\theta<\alpha \quad (5.4.8) \\ N(\theta), & \beta<\theta<\pi \quad (5.4.9) \end{cases}$$

where

$$M(\theta)=g_1(\theta)-\frac{1}{2}(-)^m\int_\alpha^\beta g_2(x)S_m(\theta,x)\mathrm{Sin}x dx \qquad (5.4.10)$$

and $$N(\theta)=g_3(\theta)-\frac{1}{2}(-)^m\int_\alpha^\beta g_2(x)S_m(\theta,x)\mathrm{Sin}x dx \qquad (5.4.11)$$

Now using the summation result (5.4.5) in terms of integral in equation (5.4.8), we obtain

$$\frac{2.\frac{1}{2}(-)^{2m}}{\pi(S_1S_2)^m}\left[\int_0^\alpha k_1(x)\int_0^{\min(S_1,S_2)}\frac{S^{2m}ds}{\left[(S_1{}^2-S^2)(S_2{}^2-S^2)\right]^{1/2}}.\mathrm{Sin}x dx\right.$$

$$\left.+\int_\beta^\pi k_1(x)\int_0^{\min(S_1,S_2)}\frac{S^{2m}ds}{\left[(S_1{}^2-S^2)(S_2{}^2-S^2)\right]^{1/2}}\mathrm{Sin}x dx\right]$$

$$=M(\theta), \qquad 0\le\theta<\alpha \qquad (5.4.12)$$

$$\frac{\operatorname{Sin}^{-m}\theta}{\pi}\left[\int_0^{\theta} k_1(x)\operatorname{Sin}^{1-m}x\,dx\int_0^{S_2}\frac{S^{2m}ds}{\left[\left(S_1^{\,2}-S^2\right)\left(S_2^{\,2}-S^2\right)\right]^{1/2}}.\right.$$

$$+\int_{\theta}^{\alpha} k_1(x)\operatorname{Sin}^{1-m}x\,dx\int_0^{S_1}\frac{S^{2m}ds}{\left[\left(S_1^{\,2}-S^2\right)\left(S_2^{\,2}-S^2\right)\right]^{1/2}}$$

$$\left.+\int_{\beta}^{\pi} k_2(x)\operatorname{Sin}^{1-m}x\,dx\int_0^{S_1}\frac{S^{2m}ds}{\left[\left(S_1^{\,2}-S^2\right)\left(S_2^{\,2}-S^2\right)\right]^{1/2}}\right]$$

$$= M(\theta) \qquad 0\le\theta<\alpha \qquad (5.4.13)$$

Now changing the variables by putting the values of S_1, S_2 and S in equation (5.4.14), we get

$$\frac{\operatorname{Sin}^{-m}\theta}{\pi}\left[\int_0^{\theta} k_1(x)\operatorname{Sin}^{1-m}x\,dx\,\frac{1}{2}\int_0^{x}\frac{\left(2\operatorname{Cos}\frac{\theta}{2}\operatorname{Cos}\frac{x}{2}\tan\frac{u}{2}\right)^{2m}}{\left[(\operatorname{Cos}u-\operatorname{Cos}\theta)(\operatorname{Cos}u-\operatorname{Cos}x)\right]^{1/2}}du.\right.$$

$$+\int_{\theta}^{\alpha} k_1(x)\operatorname{Sin}^{1-m}x\,dx\,\frac{1}{2}\int_0^{\theta}\frac{\left(2\operatorname{Cos}\frac{\theta}{2}\operatorname{Cos}\frac{x}{2}\tan\frac{u}{2}\right)^{2m}}{\left[(\operatorname{Cos}u-\operatorname{Cos}\theta)(\operatorname{Cos}u-\operatorname{Cos}x)\right]^{1/2}}du$$

$$\left.+\int_{\beta}^{\pi} k_2(x)\operatorname{Sin}^{1-m}x\,dx\,\frac{1}{2}\int_0^{\theta}\frac{\left(2\operatorname{Cos}\frac{\theta}{2}\operatorname{Cos}\frac{x}{2}\tan\frac{u}{2}\right)^{2m}}{\left[(\operatorname{Cos}u-\operatorname{Cos}\theta)(\operatorname{Cos}u-\operatorname{Cos}x)\right]^{1/2}}du\right]$$

$$= M(\theta), \qquad 0\le\theta<\alpha \qquad (5.4.14)$$

Inverting the order of integration, in the above equation, we have

$$\frac{2^{-m}\operatorname{Sin}^{-m}\theta/2\operatorname{Cos}^{-m}\theta/2.2^{2m}\operatorname{Cos}^{2m}\theta/2}{2\pi}\left[\int_0^\theta \frac{\tan^{2m}u/2}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du\right.$$

$$\times\int_u^\theta \frac{2^{-m}k_1(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_0^\theta \frac{\tan^{2m}u/2}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$\times\int_\theta^\alpha \frac{2^{-m}k_1(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_0^\theta \frac{\tan^{2m}u/2}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$\left.\times\int_\beta^\pi \frac{2^{-m}k_2(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx\right]=M(\theta), \qquad 0\le\theta<\alpha \qquad (5.4.15)$$

$$\frac{2^{2m}\tan^{-m}\theta/2\operatorname{Cot}^{2m}\theta/2}{2\pi}\left[\int_0^\theta \frac{\tan^{2m}u/2\,du}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}\right.$$

$$\int_u^\alpha \frac{k_1(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx+\int_0^\theta \frac{\tan^{2m}u/2}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du$$

$$\left.\int_\beta^\pi \frac{k_2(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx\right]=M(\theta), \qquad 0\le\theta<\alpha$$

Now above equation can be written as

$$\frac{1}{2\pi}\int_0^\theta \frac{\tan^{2m}u/2}{(\operatorname{Cos}u-\operatorname{Cos}\theta)^{1/2}}du\left[\int_u^\alpha \frac{k_1(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx\right.$$

$$\left.+\int_\beta^\pi \frac{k_2(x)\operatorname{Cot}^m x/2.\operatorname{Sin}x}{(\operatorname{Cos}u-\operatorname{Cos}x)^{1/2}}dx\right]=\tan^m\frac{1}{2}\theta.M(\theta), \qquad 0\le\theta<\alpha \qquad (5.4.16)$$

Equation (3.8.16) is now reduced to the following form

$$\int_0^{\theta}\frac{\tan^{2m} u/2}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du\left[J(u)+\frac{1}{2\pi}\int_{\beta}^{\pi}\frac{k_2(x)\mathrm{Cot}^m\, x/2.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx\right]$$

$$=\tan^m\frac{1}{2}\theta.M(\theta), \qquad 0\le\theta<\alpha \qquad (5.4.17)$$

where

$$J(u)=\frac{1}{2}\pi\int_u^{\alpha}\frac{k_1(x)\mathrm{Cot}^m\, x/2.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx \qquad (5.4.18)$$

Solution of the equation (5.4.17) can be given, with the help of the results (5.2.6) and (5.2.8), as

$$J(u)+\frac{1}{2\pi}\int_{\beta}^{\pi}\frac{k_2(x)\mathrm{Cot}^m\, 1/2\, x.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx=\frac{\mathrm{Cot}^{2m}\, u/2}{\pi}$$

$$\frac{d}{du}\int_0^{u}\frac{M(\theta)\tan^{2m} 1/2\,\theta.\mathrm{Sin}\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}u)^{1/2}}d\theta \qquad 0\le u<\alpha \qquad (5.4.19)$$

Now equation (5.4.19) becomes

$$M_1(u)=J(u)+\frac{1}{2\pi}\int_{\beta}^{\pi}\frac{k_2(x)\mathrm{Cot}^m\, x/2\ \mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx, \qquad 0\le u<\alpha \qquad (5.4.20)$$

where

$$M_1(u)=\frac{\mathrm{Cot}^{2m}\, u/2}{\pi}\frac{d}{du}\int_0^{u}\frac{M(\theta)\tan^{m} 1/2\,\theta.\mathrm{Sin}\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}u)^{1/2}}d\theta \qquad (5.4.21)$$

Again starting from equation (5.4.9), we have

$$\frac{1}{2}(-)^m\left[\int_0^\alpha k_1(x)+\int_\beta^\pi k_2(x)\right]S_m(\theta,x)\mathrm{Sin}x dx = N(\theta),\ \beta<\theta<\pi$$

Using summation result in terms of integral by equation (5.2.5), we get

$$\frac{2.\frac{1}{2}(-)^{2m}}{\pi(S_1S_2)^m}\left[\int_0^\alpha k_1(x)\int_0^{\min(S_1,S_2)}\frac{S^{2m}ds}{\left[(S_1^2-S^2)(S_2^2-S^2)\right]^{1/2}}\mathrm{Sin}x dx\right.$$
$$\left.+\int_\beta^\pi k_2(x)\int_0^{\min(S_1,S_2)}\frac{S^{2m}ds}{\left[(S_1^2-S^2)(S_2-S^2)\right]^{1/2}}\mathrm{Sin}x dx\right] = N(\theta),\ \beta<\theta<\pi$$

$$\frac{\mathrm{Sin}^{-m}\theta}{\pi}\left[\int_0^\alpha k_1(x)\mathrm{Sin}^{1-m}x dx\int_0^{S_2}\frac{S^{2m}ds}{\left[(S_1^2-S^2)(S_2^2-S^2)\right]^{1/2}}\right.$$
$$\left.+\int_\theta^\pi k_2(x)\mathrm{Sin}^{1-m}x dx\int_0^{S_1}\frac{S^{2m}ds}{\left[(S_1^2-S^2)(S_2^2-S^2)\right]^{1/2}}\right] = N(\theta),\ \beta<\theta<\pi \qquad (5.4.22)$$

Now making change of variable in equation (5.4.22)

$$\frac{\mathrm{Sin}^{-m}\theta}{\pi}\left[\int_0^\alpha k_1(x)\mathrm{Sin}^{1-m}x dx.\frac{1}{2}\int_0^x\frac{(2\mathrm{Cos}\,\theta/2\,\mathrm{Cos}\,x/2\tan u/2)^{2m}}{\left[(\mathrm{Cos}u-\mathrm{Cos}\theta)(\mathrm{Cos}u-\mathrm{Cos}x)\right]^{1/2}}du.\right.$$
$$+\int_\beta^\theta k_2(x)\mathrm{Sin}^{1-m}x dx.\frac{1}{2}\int_0^x\frac{(2\mathrm{Cos}\,\theta/2.\mathrm{Cos}\,x/2.\tan u/2)^{2m}}{\left[(\mathrm{Cos}u-\mathrm{Cos}\theta)(\mathrm{Cos}u-\mathrm{Cos}x)\right]^{1/2}}du$$
$$\left.+\int_\theta^\pi k_2(x)\mathrm{Sin}^{1-m}x dx.\frac{1}{2}\int_0^\theta\frac{(2\mathrm{Cos}\,\theta/2.\mathrm{Cos}\,x/2.\tan u/2)^{2m}}{\left[(\mathrm{Cos}u-\mathrm{Cos}\theta)(\mathrm{Cos}u-\mathrm{Cos}x)\right]^{1/2}}du\right]$$
$$= N(\theta),\qquad \beta<\theta<\pi \qquad (5.4.23)$$

Inverting the order of integration in equation (3.4.23), we get

$$\frac{2^{-m}\,\mathrm{Sin}^{-m}\,\theta/2\,\mathrm{Cos}^{-m}\,\theta/2.2^{2m}\,\mathrm{Cos}^{2m}\,\theta/2}{2\pi}\left[\int_0^{\alpha}\frac{\tan^{2m}u/2}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du\right.$$

$$\times\int_u^{\alpha}\frac{2^{-m}k_1(x)\mathrm{Cot}^m\,x/2.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx+\int_0^{\beta}\frac{\tan^{2m}u/2}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du$$

$$\times\int_{\beta}^{\pi}\frac{2^{-m}k_2(x)\mathrm{Cot}^m\,x/2.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx+\int_{\beta}^{\theta}\frac{\tan^{2m}u/2}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du$$

$$\left.\times\int_u^{\pi}\frac{2^{-m}k_2(x)\mathrm{Cot}^m\,x/2.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx\right]=N(\theta),\qquad \beta<\theta<\pi \qquad (5.4.24)$$

$$\frac{1}{2\pi}\int_{\beta}^{\theta}\frac{\tan^{2m}1/2u\,du}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}\int_u^{\pi}\frac{k_2(x)\mathrm{Cot}^m\,1/2\,x.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx$$

$$=\tan^m\frac{1}{2}\theta.N(\theta)-\frac{1}{2\pi}\int_0^{\alpha}\frac{\tan^{2m}1/2\,u}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du$$

$$\times\int_u^{\alpha}\frac{k_1(x)\mathrm{Cot}^m\frac{1}{2}x.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx-\frac{1}{2\pi}\int_0^{\beta}\frac{\tan^{2m}\frac{1}{2}u}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du$$

$$\times\int_{\beta}^{\pi}\frac{k_2(x)\mathrm{Cot}^m\,1/2\,x.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx,\qquad \beta<\theta<\pi \qquad (5.4.25)$$

Equation (3.4.25) now becomes

$$\int_{\beta}^{\theta}\frac{K(u)\tan^{2m}\frac{1}{2}u}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du=\tan^m\frac{1}{2}\theta.N(\theta)-\int_0^{\alpha}\frac{J(u)\tan^{2m}\frac{1}{2}u}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du$$

$$-\frac{1}{2\pi}\int_0^{\beta}\frac{\tan^{2m}\frac{1}{2}u}{(\mathrm{Cos}u-\mathrm{Cos}\theta)^{1/2}}du\times\int_{\beta}^{\pi}\frac{k_2(x)\mathrm{Cot}^m\frac{1}{2}x.\mathrm{Sin}x}{(\mathrm{Cos}u-\mathrm{Cos}x)^{1/2}}dx,\beta<\theta<\pi \qquad (5.4.26)$$

where

$$K(u) = \frac{1}{2\pi}\int_u^{\pi} \frac{k_2(x)\text{Cot}^m \frac{1}{2}x.\text{Sinx}}{(\text{Cosu} - \text{Cosx})^{1/2}} dx, \tag{5.4.27}$$

Now putting the value of J(u) from (5.4.20) in equation (5.4.26), we get

$$\int_\beta^\theta \frac{K(u)\tan^{2m}\frac{1}{2}u}{(\text{Cosu} - \text{Cos}\theta)^{1/2}} du = \tan^m \frac{1}{2}\theta.N(\theta) - \int_0^\alpha \frac{\tan^{2m}\frac{1}{2}u}{(\text{Cosu} - \text{Cos}\theta)^{1/2}} du$$

$$\left\{ M_1(u) - \frac{1}{2\pi}\int_\beta^\pi \frac{k_2(x)\text{Cot}^m \frac{1}{2}x.\text{Sinx}}{(\text{Cosu} - \text{Cosx})^{1/2}} dx, \right\}$$

$$-\frac{1}{2\pi}\int_0^\beta \frac{\tan^{2m}\frac{1}{2}u}{(\text{Cosu} - \text{Cos}\theta)^{1/2}} du \int_\beta^\pi \frac{k_2(x)\text{Cot}^m \frac{1}{2}x.\text{Sinx}}{(\text{Cosu} - \text{Cosx})^{1/2}} dx, \beta < \theta < \pi \tag{5.4.28}$$

$$\int_\beta^\theta \frac{K(u)\tan^{2m}\frac{1}{2}u}{(\text{Cosu} - \text{Cos}\theta)^{1/2}} du = \tan^m \frac{1}{2}\theta.N(\theta) - \int_0^\alpha \frac{M_1(u)\tan^{2m} 1/2\, u}{(\text{Cosu} - \text{Cos}\theta)^{1/2}} du$$

$$-\frac{1}{2\pi}\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}u}{(\text{Cosu} - \text{Cos}\theta)^{1/2}} du \int_\beta^\pi \frac{k_2(x)\text{Cot}^m 1/2\, x.\text{Sinx}}{(\text{Cosu} - \text{Cosx})^{1/2}} dx, \ \beta < \theta < \pi \tag{5.4.29}$$

In view of the results (5.2.6) (5.2.8), equation (5.4.29) gives

$$K(u)=\frac{\operatorname{Cot}^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u\frac{\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}\theta-\operatorname{Cos}u)^{1/2}}\left[\tan^m\frac{1}{2}\theta.N(\theta)\right.$$

$$-\int_0^\alpha\frac{M_1(\vartheta)\tan^{2m}\frac{1}{2}\vartheta}{(\operatorname{Cos}\nu-\operatorname{Cos}\theta)^{1/2}}d\vartheta-\int_\alpha^\beta\frac{\tan^{2m}\frac{1}{2}\vartheta}{(\operatorname{Cos}\nu-\operatorname{Cos}\theta)^{1/2}}d\vartheta,$$

$$\left.\times\int_\beta^\pi\frac{k_2(x)\operatorname{Cot}^m\frac{1}{2}x.\operatorname{Sin}x}{(\operatorname{Cos}\vartheta-\operatorname{Cos}x)^{1/2}}dx\right],\qquad \beta<u<\pi \tag{5.4.30}$$

Equation (5.4.30) is now reduced to

$$K(u)=N_1(u)+M_2(u)-\frac{\operatorname{Cot}^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u\frac{\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}\theta-\operatorname{Cos}u)^{1/2}}$$

$$\times\int_\alpha^\beta\frac{\tan^{2m}\frac{1}{2}\vartheta d\vartheta}{(\operatorname{Cos}\vartheta-\operatorname{Cos}x)^{1/2}}\int_\beta^\pi\frac{k_2(x)\operatorname{Cot}^m\frac{1}{2}x.\operatorname{Sin}x}{(\operatorname{Cos}\vartheta-\operatorname{Cos}x)^{1/2}}dx,\quad \beta<u<\pi \tag{5.4.31}$$

$$N_1(u)=\frac{\operatorname{Cot}^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u\frac{N(\theta)\tan^m\frac{1}{2}\theta.\operatorname{Sin}\theta}{(\operatorname{Cos}\theta-\operatorname{Cos}u)^{1/2}}d\theta \tag{5.4.32}$$

and

$$M_2(u)=\frac{-\operatorname{Cot}^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u\frac{\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}\theta-\operatorname{Cos}u)^{1/2}}$$

$$\int_0^{\alpha} \frac{M_1(\vartheta)\tan^{2m}\frac{1}{2}\vartheta}{(\text{Cos}\vartheta - \text{Cos}x)^{1/2}} d\vartheta \tag{5.4.33}$$

Changing the order of integration of the equation (5.4.3), we get

$$K(u) = N_1(u) + M_2(u) - \frac{\text{Cot}^{2m}\frac{1}{2}u}{2\pi^2}\left[\int_{\alpha}^{\beta} \tan^{2m}\frac{1}{2}\vartheta d\vartheta\right.$$

$$\left.\frac{d}{du}\int_{\beta}^{u} \frac{\text{Sin}\theta d\theta}{(\text{Cos}\vartheta - \text{Cos}\theta)^{1/2}(\text{Cos}\theta - \text{Cos}u)^{1/2}} \int_{\beta}^{\pi} \frac{k_2(x)\text{Cot}^m\frac{1}{2}x.\text{Sin}x}{(\text{Cos}\vartheta - \text{Cos}x)^{1/2}} dx,\right]$$

$$\beta < u < \pi \tag{5.4.34}$$

using the result given by

$$\frac{d}{du}\int_{\beta}^{u} \frac{\text{Sin}\theta d\theta}{(\text{Cos}\vartheta - \text{Cos}\theta)^{1/2}(\text{Cos}\theta - \text{Cos}u)^{1/2}}$$

$$= \frac{\text{Sin}u(\text{Cos}\vartheta - \text{Cos}\beta)^{1/2}}{(\text{Cos}\beta - \text{Cos}u)^{1/2}(\text{Cos}\vartheta - \text{Cos}u)} \tag{5.4.35}$$

In equation (5.4.34), we get

$$K(u) = N_1(u) + M_2(u) - \frac{\text{Cot}^{2m}\frac{1}{2}u\text{Sin}u}{2\pi^2(\text{Cos}\beta - \text{Cos}u)^{1/2}}\left[\int_{\alpha}^{\beta} \frac{\tan^{2m}\frac{1}{2}\vartheta(\text{Cos}\vartheta - \text{Cos}\beta)^{1/2}}{(\text{Cos}\vartheta - \text{Cos}u)} dv\right.$$

$$\times \int_{\beta}^{\pi} \frac{k_2(x) \text{Cot}^m \frac{1}{2} x.\text{Sinx}}{(\text{Cos}\vartheta - \text{Cosx})^{1/2}} dx, \Bigg], \qquad \beta < u < \pi \tag{5.4.36}$$

Now with the help of the results (5.2.7) and (5.2.9), the solution of the equation (5.4.27) is given by

$$k_2(x).\text{Cot}^m \frac{1}{2} x.\text{Sinx} = -2 \frac{d}{dx} \int_{x}^{\pi} \frac{K(u)\text{Sinu}}{(\text{Cosx} - \text{Cosu})^{1/2}} du \tag{5.4.37}$$

By (3.8.37) we obtain

$$\int_{\beta}^{\pi} \frac{k_2(x) \text{Cot}^m \frac{1}{2} x.\text{Sinx}}{(\text{Cos}\vartheta - \text{Cosx})^{1/2}} dy = \frac{2}{(\text{Cos}\vartheta - \text{Cos}\beta)^{1/2}}$$

$$\times \int_{\beta}^{\pi} \frac{K(s)\text{Sinsds}}{(\text{Cos}\beta - \text{Coss})^{1/2} (\text{Cos}\vartheta - \text{Coss})} \tag{5.4.38}$$

Putting the value form (5.4.38) in equation (5.4.36), we get

$$K(u) = N_1(u) + M_2(u) - \frac{\text{Cot}^{2m} \frac{1}{2} u\text{Sinu}}{\pi^2 (\text{Cos}\beta - \text{Cosu})^{1/2}} \int_{\alpha}^{\beta} \frac{\tan^{2m} \frac{1}{2} \vartheta (\text{Cos}\vartheta - \text{Cos}\beta)^{1/2}}{(\text{Cos}\vartheta - \text{Cosu})}$$

$$\times \frac{1}{(\text{Cos}\vartheta - \text{Cos}\beta)^{1/2}} \int_{\beta}^{\pi} \frac{k(s).\text{Sinsds}}{(\text{Cos}\beta - \text{Coss})^{1/2} (\text{Cos}\vartheta - \text{Cosx})}, \beta < u < \pi \tag{5.4.39}$$

Changing the order of integration of equation (5.4.39), we get

$$K(u) = N_1(u) + M_2(u) - \frac{\text{Cot}^{2m}\frac{1}{2}u}{\pi^2}\int_{\beta}^{\pi}\frac{k(s).\text{Sinsds}}{(\text{Cos}\beta - \text{Cosu})^{1/2}(\text{Cos}\beta - \text{Coss})^{1/2}}$$

$$\times\int_{\alpha}^{\beta}\frac{\tan^{2m}\frac{1}{2}\vartheta d\vartheta}{(\text{Cos}\vartheta - \text{Coss})(\text{Cos}\vartheta - \text{Coss})},$$

$$\times\int_{\alpha}^{\beta}\frac{\tan^{2m}\frac{1}{2}\vartheta d\vartheta}{(\text{Cos}\vartheta - \text{Coss})(\text{Cos}\vartheta - \text{Coss})}, \qquad \beta < u < \pi \qquad (5.4.40)$$

Equation (5.4.40) can now be rewritten as

$$K(u) + \frac{1}{\pi^2}\int_{\beta}^{\pi}K(s)R(s,u)ds = N_1(u) + M_2(u) \qquad \beta < u < \pi \qquad (5.4.41)$$

where R(s, u) is symmetric kernel,

$$R(s,u) = -\frac{\text{Cot}^{2m}\frac{1}{2}u.\text{Sins.Sinu}}{(\text{Cos}\beta - \text{Coss})^{1/2}(\text{Cos}\beta - \text{Cosu})^{1/2}}S(s,u) \qquad (5.4.42)$$

and

$$S(s,u) = \int_{\alpha}^{\beta}\frac{\tan^{2m}\frac{1}{2}\vartheta d\vartheta}{(\text{Cos}\vartheta - \text{Coss})(\text{Cos}\vartheta - \text{Cosu})} \qquad (5.4.43)$$

Equation (5.4.41) is a Fredholm integral equation of the second kind which determines K(u) and from equation (5.4.37) $k_2(x)$ can be find. After that, we can calculate the value of J(u) from equation (5.4.20) and consequently $k_1(x)$ can be found

with the help of equation (5.4.18). Finally, the unknown coefficients B_n can be computed with the help of the equation (5.3.2) which satisfy the equations (5.1.4), (5.1.5) and (5.1.6).

PARTICULAR CASE

If we let $\beta = \pi$, the equation from (5.1.4) to (5.1.6) reduce to the dual series equations and the solution obtained here agree with that obtained earlier by Collins [10].

CHAPTER –6

FIVE SERIES EQUATION

6. INTRODUCTION

There has been a lot of work on dual, triple and quadruple series equations involving different polynomials. Due to the importance of these series in finding the solutions of various mixed boundary value problems of elasticity, electrostatics and other fields of mathematical physics, a number of researchers took interest in finding the series solution as well as developing and investigating new classes of series equations.

There was almost no research work on five series equations until Dwivedi and Shukla [32] taken it into consideration. They solved certain five series equations involving generalized Bateman K-functions, series of Jacobi and Laguerre and the product of 'r' generalized Bateman K–function. In the subsequent years Dwivedi and Singh [33], obtained the solution of five series equations involving generalized Bateman K–function and Jacobi polynomials respectively.

In the present chapter, we have considered five series equations involving series of Jacobi polynomials, which were untouched till date.

6.1 FIVE SERIES EQUATIONS INVOLVING SERIES OF JACOBI POLYNOMIALS

The solution of five series equations involving series of Jacobi

polynomials is obtained by reducing them to Fredholm integral equations of the second kind in one independent variable. Dual, triple, quadruple and five series equations involving series of Jacobi polynomials can be change into the series involving ultraspherical polynomials or Fourier cosine series by small amendment to the original one. These latter series equations play an important role in solving the mixed boundary value problems, when we consider the distribution of stresses in the interior of an infinitely long strips containing three Griffith cracks situated on a line perpendicular to the boundary lines of the strip.

Here we are concerned only with five series equations involving series of Jacobi polynomials which are extensions of quadruple series considered by Dwivedi and Singh.

6.2 THE EQUATIONS

We shall solve the following set of five series equations

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(n+\alpha+1)\Gamma(n+\beta+3/2)}P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta)=\begin{cases} f_1(\theta), & 0\le\theta<a \quad (6.2.1)\\ f_3(\theta), & b<\theta<c \quad (6.2.2)\\ f_5(\theta), & d<\theta<\pi \quad (6.2.3)\end{cases}$$

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(n+\beta+1)\Gamma(n+\alpha+1/2)}P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta)=\begin{cases} f_2(\theta), & a<\theta<b \quad (6.2.4)\\ f_4(\theta), & c<\theta<d \quad (6.2.5)\end{cases}$$

where α, $\beta > -\frac{1}{2}$, and $f_i(\theta)$, $(i = 1,2,3,4,5)$ are prescribed functions and equations (6.2.1) to (6.2.5) are to be solved for unknown co-efficients A_n. It is

assumed that series (6.2.1) to (6.2.5) are uniformly convergent and $f_i(\theta)$ and their derivatives are continuous.

6.3 PRELIMINARY RESULTS

In the course of analysis, we require following results:

(i) ***The Orthogonality Relation for Jacobi Polynomials***

$$\int_0^{\pi} (\operatorname{Sin}\theta/2)^{2\alpha} (\operatorname{Cos}\theta/2)^{2\beta} P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta) P_m^{(\alpha,\beta)}(\operatorname{Cos}\theta)\operatorname{Sin}\theta d\theta$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{q_n^{(\alpha,\beta)}}\delta_{mn} \qquad (6.3.1)$$

Is valid for $\alpha > -1$, $\beta > -1$

Where δ_{mn} is Kronecker delta and

$$q_n^{(\alpha+\beta)} = \frac{1}{2} n! \{(\alpha+\beta+2n+1)\}\Gamma(\alpha+\beta+n+1)$$

(ii) ***The Series***

$$S(u,\theta) = \sum_{n=0}^{\infty} = \frac{q_n^{(\alpha,\beta)}\Gamma(n+\alpha+1/2)}{\{\Gamma(n+\alpha+1)\}^2\Gamma(n+\beta+3/2)}$$

$$(\operatorname{Sin} u/2)^{2\alpha} P_n^{(\alpha,\beta)}(\operatorname{Cos}u) \times P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta) \qquad (6.3.2)$$

$$= \frac{(\operatorname{Sin}\theta/2)^{-2\alpha}}{\pi}\int_0^{\min(u,\theta)} \frac{E(y)dy}{(\operatorname{Cos}y - \operatorname{Cos}u)^{1/2}(\operatorname{Cos}y - \operatorname{Cos}\theta)^{1/2}} \qquad (6.3.3)$$

where

$$E(t) = (\text{Sin } t/2)^{2\alpha} \ (\text{Cos } t/2)^{-2\beta}, \qquad t = \min(u, \theta).$$

(iii) ***We shall use the following two forms of Schlomilch's Integral Equations:***

If $f(\theta)$ and $f'(\theta)$ are continuous in $a \le \theta \le b$, then the solutions of the integral equations:

$$f(\theta) = \int_a^\theta \frac{g(u)du}{(\text{Cos}u - \text{Cos}\theta)^{1/2}} \tag{6.3.4}$$

and
$$f'(\theta) = \int_\theta^b \frac{g'(u)du}{(\text{Cos}\theta - \text{Cos}u)^{1/2}} \tag{6.3.5}$$

are
$$g(u) = \frac{1}{\pi}\frac{d}{du}\int_a^u \frac{f(\theta)\text{Sin}\theta d\theta}{(\text{Cos}\theta - \text{Cos}u)^{1/2}} \tag{6.3.6}$$

and
$$g'(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^b \frac{f'(\theta)\text{Sin}\theta d\theta}{(\text{Cos}u - \text{Cos}\theta)^{1/2}} \tag{6.3.7}$$

respectively.

6.4 THE SOLUTION

Let us suppose

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\beta+1)\Gamma(n+\alpha+1/2)} P_n^{(\alpha,\beta)}(\text{Cos}\theta) = \begin{cases} g(\theta), & 0 \le \theta < a \quad (6.4.1) \\ h(\theta), & b < \theta < c \quad (6.4.2) \\ k(\theta), & d < \theta < \pi \quad (6.4.3) \end{cases}$$

Using orthogonality relation (6.3.1) in equations (6.2.4), (6.2.5) and (6.4.1) to (6.4.3) we get

$$A_n = \frac{\Gamma(n+\alpha+1/2)}{\Gamma(n+\alpha+1)} q_n^{(\alpha,\beta)} \left[\int_0^a g'(u) + \int_a^b f'_2(u) + \int_b^c h'(u) \times \int_c^d f'_4(u) \right.$$

$$\left. + \int_d^{\pi} k'(\theta) (\operatorname{Sin} u/2)^{2\alpha} P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta) \times P_n^{(\alpha,\beta)}(\operatorname{Cos} u) du \right. \qquad (6.4.4)$$

where

$g'(u) = (\operatorname{Cos} u/2)^{2\beta} \operatorname{Sin} u\, g(u)$,

$f_2'(u) = (\operatorname{Cos} u/2)^{2\beta} \operatorname{Sin} u\, f_2(u)$, etc.

Substituting the expression for A_n from (6.4.4) in equations (6.2.1) to (6.2.3), we obtain

$$\sum_{n=0}^{\infty} \frac{q_n^{(\alpha,\beta)} \Gamma(n+\alpha+1/2)}{\{\Gamma(n+\alpha+1)\}^2 \Gamma(n+\beta+3/2)} \left[\int_0^a g'(u) + \int_a^b f'_2(u) + \int_b^c h'(u) \times \int_c^d f'_4(u) \right.$$

$$\left. + \int_d^{\pi} k'(u) \right] (\operatorname{Sin} u/2)^{2\alpha} P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta) P_n^{(\alpha,\beta)}(\operatorname{Cos} u) du$$

$$= \begin{cases} f_1(\theta) & 0 \le \theta < a \qquad (6.4.5) \\ f_3(\theta) & b < \theta < c \qquad (6.4.6) \\ f_5(\theta) & d < \theta < \pi \qquad (6.4.7) \end{cases}$$

Applying the summation result (6.3.2) and interchanging the order of integration and summation, we get

$$\left[\int_0^a g'(u)+\int_b^c h'(u)+\int_d^\pi k'(u)\right]S(u,\theta)du\begin{cases}P(\theta), & 0\le\theta<a \quad (6.4.8)\\ Q(\theta), & b<\theta<c \quad (6.4.9)\\ R(\theta), & d<\theta<\pi \quad (6.4.10)\end{cases}$$

where

$$P(\theta)=f_1(\theta)-\left[\int_a^b f'_2(u)+\int_c^d f'_4(u)\right]S(u,\theta)du$$

$$Q(\theta)=f_3(\theta)-\left[\int_a^b f'_2(u)+\int_c^d f'_4(u)\right]S(u,\theta)du$$

and

$$R(\theta)=f_5(\theta)-\left[\int_a^b f'_2(u)+\int_c^d f'_4(u)\right]S(u,\theta)du$$

Now using the summation result (6.3.3) in terms of integral in equation (6.4.8) we obtain

$$\int_0^a g'(u)du\int_0^{\min(u,\theta)}\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cos\theta)^{1/2}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_b^c h'(u)du\int_0^{\min(u,\theta)}\frac{E(y)dy}{(Cosy-Cosu)^{\frac{1}{2}}(Cosy-Cosu)^{\frac{1}{2}}}\right.$$

$$\left.+\int_d^\pi k'(u)du\int_0^{\min(u,\theta)}\frac{E(y)dy}{(Cosy-Cosu)^{\frac{1}{2}}(Cosy-Cosu)^{\frac{1}{2}}}\right]\quad 0\le\theta<a \quad (6.4.11)$$

or $$\int_0^\theta g'udu\int_0^u\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cos\theta)^{1/2}}$$

$$+\int_0^a g'(u)du\int_0^\theta\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cos\theta)^{1/2}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_b^c h'(u)du\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$\left.+\int_d^\pi k'(u)du\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right] \qquad 0\le\theta<a \qquad (6.4.12)$$

Changing the order of integration in the above equation, we get

$$\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_y^\theta \frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}+\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}$$

$$\times\int_\theta^a \frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}=\pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_b^c \frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$\left.+\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_d^\pi \frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right] \qquad 0\le\theta<a \qquad (6.4.13)$$

or $$\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_y^a \frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}=\pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_b^c \frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$\left.+\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_d^\pi \frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right] \qquad 0\le\theta<a \qquad (6.4.14)$$

Using results (6.3.4) and (6.3.6) in eq. (6.4.14), we obtain

$$E(y)\int_y^a \frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}} = \pi\frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} P(\theta)\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}}\left\{\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_b^c \frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$\left.+\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_d^\pi \frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right\} \qquad 0 \le \theta < a \qquad (6.4.15)$$

Changing the order of integration of above equation, we get

$$E(y)\int_y^a \frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}} = \frac{d}{dy}\int_0^y \frac{\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} p(\theta)\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\left\{\int_b^c \frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\frac{d}{dy}\int_0^y E(y)dy\int_t^y \frac{\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}(\mathrm{Cos}t-\mathrm{Cos}\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\int_d^\pi \frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\frac{d}{dy}\int_0^y E(y)dy\int_t^y \frac{\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}(\mathrm{Cos}t-\mathrm{Cos}\theta)^{\frac{1}{2}}}\right\}$$

$$0 \le \theta < a \qquad (6.4.16)$$

Using the result

$$\int_t^y \frac{\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}(\mathrm{Cos}t-\mathrm{Cos}\theta)^{\frac{1}{2}}} = \pi \qquad (6.4.17)$$

in equation (6.4.16), we get

$$E(y)\int_y^{\alpha}\frac{g'(u)du}{(\cos y-\cos u)^{\frac{1}{2}}}=P_1(y)-\int_b^c\frac{E(y)h'(u)du}{(\cos y-\cos u)^{\frac{1}{2}}}$$

$$-\int_d^{\pi}\frac{E(y)k'(u)du}{(\cos y-\cos u)^{\frac{1}{2}}} \qquad 0\le\theta<a \qquad (6.4.18)$$

where

$$P_1(y)=\frac{d}{dy}\int_0^y\frac{\left(\sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)\sin\theta d\theta}{(\cos\theta-\cos y)^{\frac{1}{2}}} \qquad (6.4.19)$$

Again using the results (6.3.5) and (6.3.7) in equation (6.4.10), we get

$$E(y)g'(u)=-\frac{1}{\pi}\frac{d}{du}\int_u^a\frac{\sin y P_1(y)dy}{(\cos u-\cos y)^{\frac{1}{2}}}+\frac{1}{\pi}\frac{d}{du}\int_u^a\frac{E(y)\sin y dy}{(\cos u-\cos y)^{\frac{1}{2}}}$$

$$+\int_b^e\frac{h'(v)dv}{(\cos y-\cos v)^{\frac{1}{2}}}+\frac{1}{\pi}\frac{d}{du}\int_u^a\frac{E(y)\sin y dy}{(\cos u-\cos y)^{\frac{1}{2}}}$$

$$\text{x}\int_d^{\pi}\frac{k'(v)dv}{(\cos y-\cos v)^{\frac{1}{2}}} \qquad 0\le u<a \qquad (6.4.20)$$

Now changing the order of integration and using the result

$$\frac{d}{du}\int_a^u\frac{\sin y dy}{(\cos u-\cos y)^{\frac{1}{2}}(\cos y-\cos v)^{\frac{1}{2}}}=\frac{\sin u(\cos v-\cos a)^{\frac{1}{2}}}{(\cos a-\cos u)^{\frac{1}{2}}(\cos v-\cos u)^{\frac{1}{2}}} \qquad (6.4.21)$$

in equation (6.4.20), becomes

$$E(y)g'(u) = P_2(u) + \frac{1}{\pi}\left[\int_b^c h'(v)A(v,y)dv + \int_d^{\pi} k'(v)A(v,y)dv\right]$$

$$0 < u < a \qquad (6.4.22)$$

where

$$P_2(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^a \frac{\text{Sin}yP_1(y)dy}{(\text{Cos}u - \text{Cos}y)^{\frac{1}{2}}} \qquad (6.4.23)$$

$$A(v,y) = \frac{E(y)\text{Sin}u(\text{Cos}v - \text{Cos}a)^{\frac{1}{2}}}{(\text{Cos}a - \text{Cos}u)^{\frac{1}{2}}(\text{Cos}v - \text{Cos}u)} \qquad (6.4.24)$$

Again using summation result (6.3.3), in terms of integral in equation (6.4.9), we obtain

$$\left[\int_b^{\theta} h'(u) + \int_{\theta}^{c} h'(u)\right]\int_0^{\min(u,\theta)} \frac{E(y)dudy}{(\text{Cos}y - \text{Cos}u)(\text{Cos}y - \text{Cos}\theta)^{1/2}}$$

$$= \pi\left(\text{Sin}\frac{\theta}{2}\right)^{2\alpha} Q(\theta) - \left[\int_0^a g'(u)du + \int_d^{\pi} k'(u)du\right]$$

$$\text{x}\int_0^{\min(u,\theta)} \frac{E(y)dy}{(\text{Cos}y - \text{Cos}u)(\text{Cos}y - \text{Cos}\theta)^{1/2}} \qquad b < \theta < c \qquad (6.4.25)$$

or

$$\int_b^\theta h'(u)du\int_0^u \frac{E(y)dy}{(Cosy-Cosu)^{\frac{1}{2}}(Cosy-Cos\theta)^{\frac{1}{2}}}$$

$$+\int_\theta^c h'(u)du\int_0^\theta \frac{E(y)dy}{(Cosy-Cosu)^{\frac{1}{2}}(Cosy-Cos\theta)^{\frac{1}{2}}}$$

$$=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}.Q(\theta)-\left[\int_0^a g'(u)du\int_0^u \frac{E(y)dy}{(Cosy-Cosu)^{\frac{1}{2}}(Cosy-Cos\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\int_d^\pi k'(u)du\int_0^\theta \frac{E(y)dy}{(Cosy-Cosu)^{\frac{1}{2}}(Cosy-Cos\theta)^{\frac{1}{2}}}\right] \qquad b<\theta<c \qquad (6.4.26)$$

Changing the order of above equation, we get

$$\int_b^\theta \frac{E(y)dy}{(Cosy-Cos\theta)^{\frac{1}{2}}}\int_y^c \frac{h'(u)du}{(Cosy-Cosu)^{\frac{1}{2}}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}.Q(\theta)$$

$$-\left[\int_0^b \frac{E(y)dy}{(Cosy-Cos\theta)^{\frac{1}{2}}}\int_b^c \frac{h'(u)du}{(Cosy-Cosu)^{\frac{1}{2}}}\right.$$

$$+\int_0^a \frac{E(y)dy}{(Cosy-Cos\theta)^{\frac{1}{2}}}\int_y^a \frac{g'(u)du}{(Cosy-Cosu)^{\frac{1}{2}}}$$

$$\left.+\int_0^\theta \frac{E(y)dy}{(Cosy-Cos\theta)^{\frac{1}{2}}}\int_d^\pi \frac{k'(u)du}{(Cosy-Cosu)^{\frac{1}{2}}}\right] \qquad b<\theta<c \qquad (6.4.27)$$

Using the results (6.3.4) and (6.3.6) in equation (6.4.27) we get

$$E(y)\int_y^c \frac{h'(u)du}{(Cosy - Cosu)^{\frac{1}{2}}} = \frac{d}{dy}\int_b^y \frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha} Q(\theta)Sin\theta d\theta}{(Cos\theta - Cosy)^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\frac{d}{dy}\left\{\int_b^y \frac{Sin\theta d\theta}{(Cos\theta - Cosy)^{\frac{1}{2}}}\left[\int_0^b \frac{E(y)dy}{(Cosy - Cos\theta)^{\frac{1}{2}}}\int_b^c \frac{h'(u)du}{(Cosy - Cosu)^{\frac{1}{2}}}\right.\right.$$

$$+\int_0^a \frac{E(y)dy}{(Cosy - Cos\theta)^{\frac{1}{2}}}\frac{g'(u)du}{(Cosy - Cosu)^{\frac{1}{2}}} + \int_0^\theta \frac{E(y)dy}{(Cosy - Cos\theta)^{\frac{1}{2}}}$$

$$\left. x\int_d^\pi \frac{k'(u)du}{(Cosy - Cosu)^{\frac{1}{2}}}\right] \qquad b < \theta < c \qquad (6.4.28)$$

Changing the order of integration in above equation, we get

$$E(y)\int_y^c \frac{h'(u)du}{(Cosy - Cosu)^{\frac{1}{2}}} = Q_1(y) - \frac{1}{\pi}\left[\int_b^c \frac{h'(u)du}{(Cosy - Cosu)^{\frac{1}{2}}}\int_0^b E(y)dy\right.$$

$$\left.+\int_t^a \frac{g'(u)du}{(Cost - Cosu)^{\frac{1}{2}}}\int_0^a E(y)dy + \int_d^\pi \frac{k'(u)du}{(Cost - Cosu)^{\frac{1}{2}}}\int_0^\theta E(y)dy\right]$$

$$x\frac{d}{dy}\int_b^y \frac{Sin\theta d\theta}{[(Cos\theta - Cosy)(Cost - Cos\theta)]^{\frac{1}{2}}} \qquad b < \theta < c \qquad (6.4.29)$$

where

$$Q_1(y) = \frac{d}{dy}\int_b^y \frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha} Q(\theta)Sin\theta d\theta}{(Cos\theta - Cosy)^{\frac{1}{2}}} \qquad (6.4.30)$$

$$\frac{d}{dy}\int_b^y \frac{\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}(\mathrm{Cos}t-\mathrm{Cos}\theta)^{\frac{1}{2}}} = \frac{\mathrm{Sin}y(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}}{(\mathrm{Cos}b-\mathrm{Cos}y)^{\frac{1}{2}}(\mathrm{Cos}t-\mathrm{Cos}y)^{\frac{1}{2}}} \qquad (6.4.31)$$

in equation (6.4.29), we get

$$E(y)\int_y^c \frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}} = Q_1(y) - \frac{1}{\pi}\frac{\mathrm{Sin}y}{(\mathrm{Cos}b-\mathrm{Cos}y)^{\frac{1}{2}}}$$

$$\left[\int_0^b \frac{E(t)(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}dt}{(\mathrm{Cos}t-\mathrm{Cos}y)}\int_b^c \frac{h'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$+\int_0^a \frac{E(t)(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}dt}{(\mathrm{Cos}t-\mathrm{Cos}y)}\int_t^a \frac{g'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}$$

$$\left.+\int_0^\theta \frac{E(t)(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}dt}{(\mathrm{Cos}t-\mathrm{Cos}y)}\int_d^\pi \frac{k'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}\right] \qquad b<y<c \qquad (6.4.32)$$

Putting the value of last integral of second term on the right hand side of eq. (6.4.32) from eq. (6.4.18), we get

$$E(y)\int_y^c \frac{h'(u)du}{(\mathrm{Cosb}-\mathrm{Cos}y)^{\frac{1}{2}}} = Q_1(y) - \frac{1}{\pi}\left[\frac{\mathrm{Sin}y}{(\mathrm{Cos}b-\mathrm{Cos}y)^{\frac{1}{2}}}\right.$$

$$\int_0^b \frac{E(t)(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}}{(\mathrm{Cos}t-\mathrm{Cos}y)}dt\int_b^c \frac{h'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}$$

$$+\int_0^a \frac{E(t)(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}}{(\mathrm{Cos}t-\mathrm{Cos}y)}dt\left\{\frac{P_1(t)}{E(t)}-\int_b^c \frac{h'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$-\int_d^\pi \frac{k'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}} + \int_0^\theta \frac{E(t)(\mathrm{Cos}t-\mathrm{Cos}b)^{\frac{1}{2}}}{(\mathrm{Cos}t-\mathrm{Cos}y)}dt$$

$$\left.\int_d^{\pi}\frac{k'(u)du}{(Cost-Cosu)^{\frac{1}{2}}}\right] \qquad b<y<c \qquad (6.4.33)$$

$$E(y)\int_y^c\frac{h'(u)du}{(Cosy-Cosu)^{\frac{1}{2}}}=Q_1(y)+P_3(y)-\frac{1}{\pi}\frac{Siny}{(Cosb-Cosy)^{\frac{1}{2}}}$$

$$\left[\int_a^b\frac{E(t)(Cost-Cosb)^{\frac{1}{2}}dt}{(Cost-Cosy)}\int_b^c\frac{h'(u)du}{(Cost-Cosu)^{\frac{1}{2}}}\right.$$

$$+\int_a^{\theta}\frac{E(t)(Cost-Cosb)^{\frac{1}{2}}dt}{(Cost-Cosy)}\int_d^{\pi}\frac{k'(u)du}{(Cost-Cosu)^{\frac{1}{2}}} \qquad b<y<c \qquad (6.4.34)$$

where

$$P_3(y)=\frac{Siny}{(Cosb-Cosy)^{\frac{1}{2}}}\int_b^a\frac{P_1(t)(Cost-Cosb)^{\frac{1}{2}}dt}{(Cost-Cosy)} \qquad (6.4.35)$$

Let $\quad B(u,y)=\dfrac{Siny}{(Cosb-Cosy)^{\frac{1}{2}}}\displaystyle\int_a^b\frac{E(t)(Cost-Cosb)^{\frac{1}{2}}dt}{(Cost-Cosy)} \qquad (6.4.36)$

and $\quad C(u,y)=\dfrac{Siny}{(Cosb-Cosy)^{\frac{1}{2}}}\displaystyle\int_a^{\theta}\frac{E(t)(Cost-Cosb)^{\frac{1}{2}}dt}{(Cost-Cosy)} \qquad (6.4.36)$

Now equation (6.4.34) can be reduced to the following form

$$E(y)\int_y^c\frac{h'(u)du}{(Cosy-Cosu)^{\frac{1}{2}}}=Q_1(y)+P_3(y)$$

$$-\frac{1}{\pi}\left[\int_b^c \frac{h'(u)B(u,y)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}+\int_d^\pi \frac{k'(u)C(u,y)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{\frac{1}{2}}}\right] \qquad b<y<c \qquad (6.4.38)$$

Again applying the results (6.3.5) and (6.3.7), in equation (6.4.38) we get

$$E(y)h'(u)=-\frac{1}{\pi}\frac{d}{du}\int_u^c \frac{\{Q_1(y)+P_3(y)\}\mathrm{Sin}s\,ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{\frac{1}{2}}}+\frac{1}{\pi^2}\frac{d}{du}\int_u^c \frac{\mathrm{Sin}s\,ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{\frac{1}{2}}}$$

$$\left[\int_b^c \frac{B(v,y)h'(v)dv}{(\mathrm{Cos}s-\mathrm{Cos}v)^{\frac{1}{2}}}+\int_d^\pi \frac{C(v,y)k'(v)dv}{(\mathrm{Cos}s-\mathrm{Cos}v)^{\frac{1}{2}}}\right] b<y<c \qquad (6.4.39)$$

Changing the order of integration and using the result

$$\frac{d}{du}\int_c^u \frac{\mathrm{Sin}s\,ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{\frac{1}{2}}(\mathrm{Cos}s-\mathrm{Cos}v)^{\frac{1}{2}}}$$

$$=\frac{\mathrm{Sin}s\,ds(\mathrm{Cos}v-\mathrm{Cos}c)^{\frac{1}{2}}}{(\mathrm{Cos}c-\mathrm{Cos}s)^{\frac{1}{2}}(\mathrm{Cos}v-\mathrm{Cos}s)} \qquad (6.4.40)$$

equation (6.4.39) can be rewritten as

$$E(y)h'(u)=Q_2(u)+\frac{1}{\pi^2}\int_b^c h'(v)B(s,v)dv+\frac{1}{\pi^2}\int_d^\pi k'(v)C(s,v)dv$$

$$b<u<c \qquad (6.4.41)$$

where,

$$Q_2(u)=-\frac{1}{\pi}\frac{d}{du}\int_u^c \frac{\{Q_1(y)+P_3(y)\}\mathrm{Sin}s\,ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{\frac{1}{2}}} \qquad (6.4.42)$$

$$B(s,v) = \frac{\operatorname{Sin}u(\operatorname{Cos}v - \operatorname{Cos}c)^{\frac{1}{2}}}{(\operatorname{Cos}c - \operatorname{Cos}s)^{\frac{1}{2}}(\operatorname{Cos}v - \operatorname{Cos}s)} B(v,y) \tag{6.4.43}$$

and $$C(s,v) = \frac{\operatorname{Sin}u(\operatorname{Cos}v - \operatorname{Cos}c)^{\frac{1}{2}}}{(\operatorname{Cos}c - \operatorname{Cos}s)^{\frac{1}{2}}(\operatorname{Cos}v - \operatorname{Cos}s)} C(v,y) \tag{6.4.44}$$

Again using summation result (6.3.3) in terms of integral in equation (6.4.10) we get

$$\left[\int_d^\theta k'(u) + \int_\theta^\pi k'(u)\right]\int_0^{\min(u,\theta)} \frac{E(y)}{[(\operatorname{Cos}y - \operatorname{Cos}u)(\operatorname{Cos}y - \operatorname{Cos}\theta)]^{1/2}} du dy$$

$$= \pi\left(\operatorname{Sin}\frac{\theta}{2}\right)^{2\alpha} R(\theta) - \left[\int_0^a g'(u)du + \int_b^c h'(u)du\right]$$

$$x\int_0^{\min(u,\theta)} \frac{E(y)dy}{(\operatorname{Cos}y - \operatorname{Cos}u)(\operatorname{Cos}y - \operatorname{Cos}\theta)^{1/2}} \qquad d < \theta < c \tag{6.4.45}$$

$$\int_d^\theta k'(u)du\int_0^u \frac{E(y)dy}{(\operatorname{Cos}y - \operatorname{Cos}u)^{\frac{1}{2}}(\operatorname{Cos}y - \operatorname{Cos}\theta)^{\frac{1}{2}}}$$

or $$+\int_\theta^\pi k'(u)du\int_0^\theta \frac{E(y)dy}{(\operatorname{Cos}y - \operatorname{Cos}u)^{\frac{1}{2}}(\operatorname{Cos}y - \operatorname{Cos}\theta)^{\frac{1}{2}}}$$

$$= \pi\left(\operatorname{Sin}\frac{\theta}{2}\right)^{2\alpha} .R(\theta) - \left[\int_0^a g'(u)du + \int_b^c h'(u)du\right]$$

$$\int_0^\theta \frac{E(y)dy}{(\operatorname{Cos}y - \operatorname{Cos}u)^{\frac{1}{2}}(\operatorname{Cos}y - \operatorname{Cos}\theta)^{\frac{1}{2}}} \qquad d < \theta < \pi \tag{6.4.46}$$

Changing the order of integration in eq. (6.4.46), we obtain

$$\int_d^{\theta}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_y^{\pi}\frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}=\pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha}.R(\theta)$$

$$-\left[\int_0^{a}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_0^{a}\frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}+\int_0^{b}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\right.$$

$$x\int_b^{c}\frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}+\int_b^{c}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_y^{c}\frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}$$

$$\left.+\int_0^{d}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_d^{\pi}\frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right]\qquad d<\theta<\pi \qquad (6.4.47)$$

Using the results (6.3.4) and 5.3.6), we get

$$E(y)\int_y^{\pi}\frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}=\frac{d}{dy}\int_d^{y}\frac{\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha}.R(\theta)\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\frac{d}{dy}\int_d^{y}\frac{\mathrm{Sin}\theta d\theta}{(\mathrm{Cos}\theta-\mathrm{Cos}y)^{\frac{1}{2}}}\left[\int_0^{a}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_y^{a}\frac{g'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right.$$

$$+\int_0^{b}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_b^{c}\frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}+\int_b^{c}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}$$

$$\left.x\int_y^{c}\frac{h'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}+\int_0^{d}\frac{E(y)dy}{(\mathrm{Cos}y-\mathrm{Cos}\theta)^{\frac{1}{2}}}\int_d^{\pi}\frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}\right]$$

$$d<y<\pi \qquad (6.4.48)$$

Changing the order of integration and applying the result.

$$\frac{d}{dy}\int_d^y \frac{\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}\theta-\operatorname{Cos}y)^{\frac{1}{2}}(\operatorname{Cos}t-\operatorname{Cos}\theta)^{\frac{1}{2}}}=\frac{\operatorname{Sin}y(\operatorname{Cos}t-\operatorname{Cos}d)^{\frac{1}{2}}}{(\operatorname{Cos}d-\operatorname{Cos}y)^{\frac{1}{2}}(\operatorname{Cos}t-\operatorname{Cos}y)^{\frac{1}{2}}} \tag{6.4.49}$$

in equation (6.4.48), we get

$$E(y)\int_y^{\pi}\frac{k'(u)du}{(\operatorname{Cos}y-\operatorname{Cos}u)^{\frac{1}{2}}}=R_1(y)-\frac{1}{\pi}\frac{\operatorname{Sin}y}{(\operatorname{Cos}d-\operatorname{Cos}y)^{\frac{1}{2}}}$$

$$\text{x}\left[\int_0^a\frac{E(t)(\operatorname{Cos}t-\operatorname{Cos}d)^{\frac{1}{2}}dt}{(\operatorname{Cos}t-\operatorname{Cos}y)}\int_t^a\frac{g'(u)du}{(\operatorname{Cos}y-\operatorname{Cos}u)^{\frac{1}{2}}}\right.$$

$$+\int_0^b\frac{E(t)(\operatorname{Cos}t-\operatorname{Cos}d)^{\frac{1}{2}}dt}{(\operatorname{Cos}t-\operatorname{Cos}y)^{\frac{1}{2}}}\int_b^c\frac{h'(u)du}{(\operatorname{Cos}t-\operatorname{Cos}u)^{\frac{1}{2}}}$$

$$+\int_t^c\frac{E(t)(\operatorname{Cos}t-\operatorname{Cos}d)^{\frac{1}{2}}dt}{(\operatorname{Cos}t-\operatorname{Cos}y)^{\frac{1}{2}}}\frac{h'(u)du}{(\operatorname{Cos}t-\operatorname{Cos}u)^{\frac{1}{2}}}$$

$$\left.+\int_0^d\frac{E(t)(\operatorname{Cos}t-\operatorname{Cos}d)^{\frac{1}{2}}dt}{(\operatorname{Cos}y-\operatorname{Cos}y)^{\frac{1}{2}}}\int_d^{\pi}\frac{k'(u)du}{(\operatorname{Cos}t-\operatorname{Cos}u)^{\frac{1}{2}}}\right] \qquad d<y<\pi \tag{6.4.50}$$

where

$$R_1(y)=\frac{d}{dy}\int_d^y\frac{\left(\operatorname{Sin}\frac{\theta}{2}\right)^{2\alpha}.R(\theta)\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}\theta-\operatorname{Cos}y)^{\frac{1}{2}}} \tag{6.4.51}$$

Now putting the values of last integrals of first and third terms on the right hand side from equations (6.4.18) and (6.4.38), we get

$$E(y)\int_y^{\pi}\frac{k'(u)du}{(\mathrm{Cos}y-\mathrm{Cos}u)^{\frac{1}{2}}}=R_1(y)+R_2(y)$$

$$-\frac{1}{\pi}\left[\int_b^c\frac{h'(u)U(u,y)du}{(\mathrm{Cost}-\mathrm{Cosu})^{\frac{1}{2}}}+\int_d^{\pi}\frac{k'(u)V(u,y)du}{(\mathrm{Cost}-\mathrm{Cosu})^{\frac{1}{2}}}\right] \qquad d<y<\pi \qquad (6.4.52)$$

where

$$R_2(y)=\frac{\sin y}{(\mathrm{Cosd}-\mathrm{Cosy})^{\frac{1}{2}}}\left[\int_0^a\frac{P_1(t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}}{(\mathrm{Cost}-\mathrm{Cosy})}dt\right.$$

$$\left.+\int_b^c\frac{\{Q_1(t)+P_3(t)\}(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}dt}{(\mathrm{Cost}-\mathrm{Cosy})}\right] \qquad (6.4.53)$$

$$U(v,y)=\frac{\sin y}{(\mathrm{Cosd}-\mathrm{Cosy})^{\frac{1}{2}}}\left[\int_0^a\frac{E(t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}}{(\mathrm{Cost}-\mathrm{Cosy})}dt\right.$$

$$\left.+\int_0^b\frac{E(t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}dt}{(\mathrm{Cost}-\mathrm{Cosy})}+\int_b^c\frac{E(t)B(v,t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}dt}{(\mathrm{Cost}-\mathrm{Cosy})}\right] \qquad (6.4.54)$$

$$V(v,y)=\frac{\sin y}{(\mathrm{Cosd}-\mathrm{Cosy})^{\frac{1}{2}}}\left[\int_0^a\frac{E(t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}dt}{(\mathrm{Cost}-\mathrm{Cosy})}\right.$$

$$\left.+\int_0^b\frac{E(t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}dt}{(\mathrm{Cost}-\mathrm{Cosy})}+\int_b^c\frac{E(t)C(v,t)(\mathrm{Cost}-\mathrm{Cosd})^{\frac{1}{2}}dt}{(\mathrm{Cost}-\mathrm{Cosy})}\right] \qquad (6.4.55)$$

Again applying the results (6.3.5) and (6.3.7) in equation (6.4.52), we get

$$E(y)h'(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^{\pi}\frac{\{R_1(y)+R_2(y)\}Sinsds}{(Cosu - Coss)^{\frac{1}{2}}}$$

$$+\frac{1}{\pi^2}\frac{d}{du}\int_u^{\pi}\frac{Sinsds}{(Cosu - Coss)^{\frac{1}{2}}}\left[\int_b^c\frac{U(v,y)h'(v)dv}{(Coss - Cosv)^{\frac{1}{2}}} + \int_d^{\pi}\frac{V(v,y)k'(v)dv}{(Coss - Cosv)^{\frac{1}{2}}}\right] \quad (6.4.56)$$

Changing the order of integration and using the result

$$\frac{d}{du}\int_{\pi}^{u}\frac{Sinsds}{(Cosu - Coss)^{\frac{1}{2}}(Coss - Cosv)^{\frac{1}{2}}} = \frac{Sinu(Cosv - Cos\pi)^{\frac{1}{2}}}{(Cos\pi - Coss)^{\frac{1}{2}}(Cosv - Coss)} \quad (6.4.57)$$

in equation (6.4.56), we get

$$E(y)k'(u) = R_3(u) + \frac{1}{\pi^2}\int_b^c h'(v)U(s,v)dv + \frac{1}{\pi^2}\int_d^{\pi} k'(v)V(s,v)dv$$

$$d < u < \pi \quad (6.4.58)$$

where,

$$R_3(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^{\pi}\frac{\{R_1(y)+R_2(y)\}Sinsds}{(Cosu - Coss)^{\frac{1}{2}}} \quad (6.4.59)$$

$$U(s,v) = \frac{Sinu(Cosv - Cos\pi)^{\frac{1}{2}}}{(Cos\pi - Coss)^{\frac{1}{2}}(Cosv - Coss)}U(v,y) \quad (6.4.60)$$

and

$$V(s,v) = \frac{Sinu(Cosv - Cos\pi)^{\frac{1}{2}}}{(Cos\pi - Coss)^{\frac{1}{2}}(Cosv - Coss)}V(v,y) \quad (6.4.61)$$

Equations (6.4.22), (6.4.41) and (6.4.58) are Fredholm integral equations of the second kind which determine $g'(u)$, $h'(u)$ $k'(u)$ respectively. Knowing the values of $g'(u)$, $h'(u)$ and $k'(u)$ the values of coefficients A_n, can be obtained from (6.4.4).

PARTICULAR CASES

If we let $d=\pi$ in equations (6.2.1) to (6.2.5), they reduce to the corresponding quadruple series equations considered earlier in [66] and the above solution agrees with that obtained previously.

Similarly, the solutions of corresponding triple and dual series can be obtained as particular cases.

CHAPTER – 7

TWO GRIFFITH-CRACKS AT THE INTERFACE OPENED BY FORCES AT CRACK SURFACES

7.1 INTRODUCTION

In the present chapter we are going to find the effect over physical quantities (important in fracture design criterion) due to the presence of two cracks in dissimilar media.

The cracks occupy the regions $y = 0$, $b<|x|<c$ at the common interface and mathematically the problem is reduced to the following boundary value problem, see figures 7.1 and 7.2.

$$\sigma_{xy}(x,0^+) = \sigma_{xy}(x,0^-) = 0 \tag{7.1.1}$$

$$\sigma_{yy}(x,0^+) = -p_1(x)$$

$$\sigma_{yy}(x,0^-) = -p_2(x) \tag{7.1.2}$$

with, $b<|x|<c$ and $(\pm)$ sign over 0 refer to the quantities corresponding to $y>0$ and $y<0$. There are the following continuity conditions.

$$u_y(x,0^+) = u_y(x,0^-), \tag{7.1.3}$$

$$u_x(x,0^+) = u_x(x,0^-), \tag{7.1.4}$$

$$\sigma_{xy}(x,0^+) = \sigma_{xy}(x,0^-) \tag{7.1.5}$$

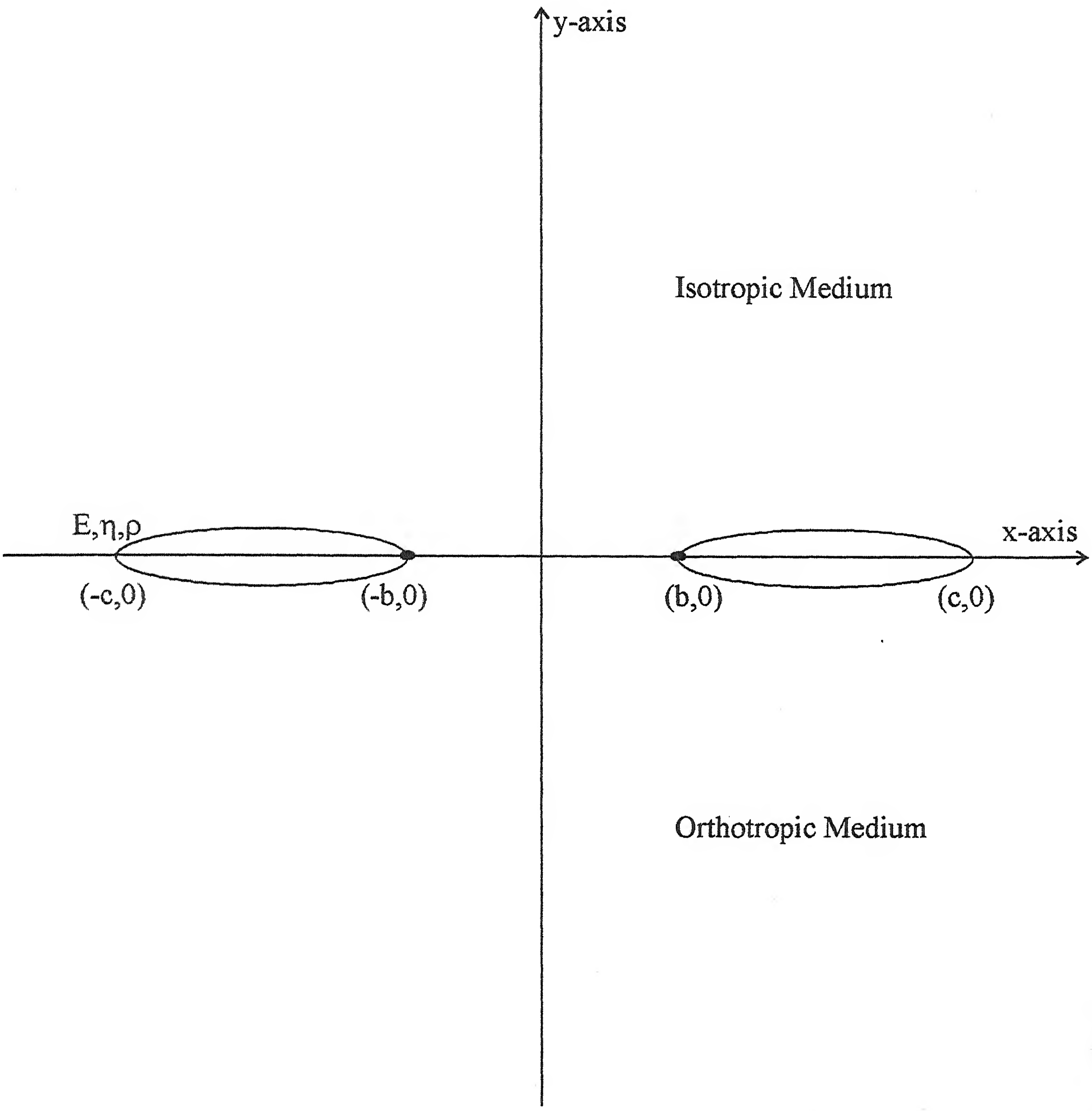

Fig. 7.1: Geometry of Problem with Two Griffith Cracks

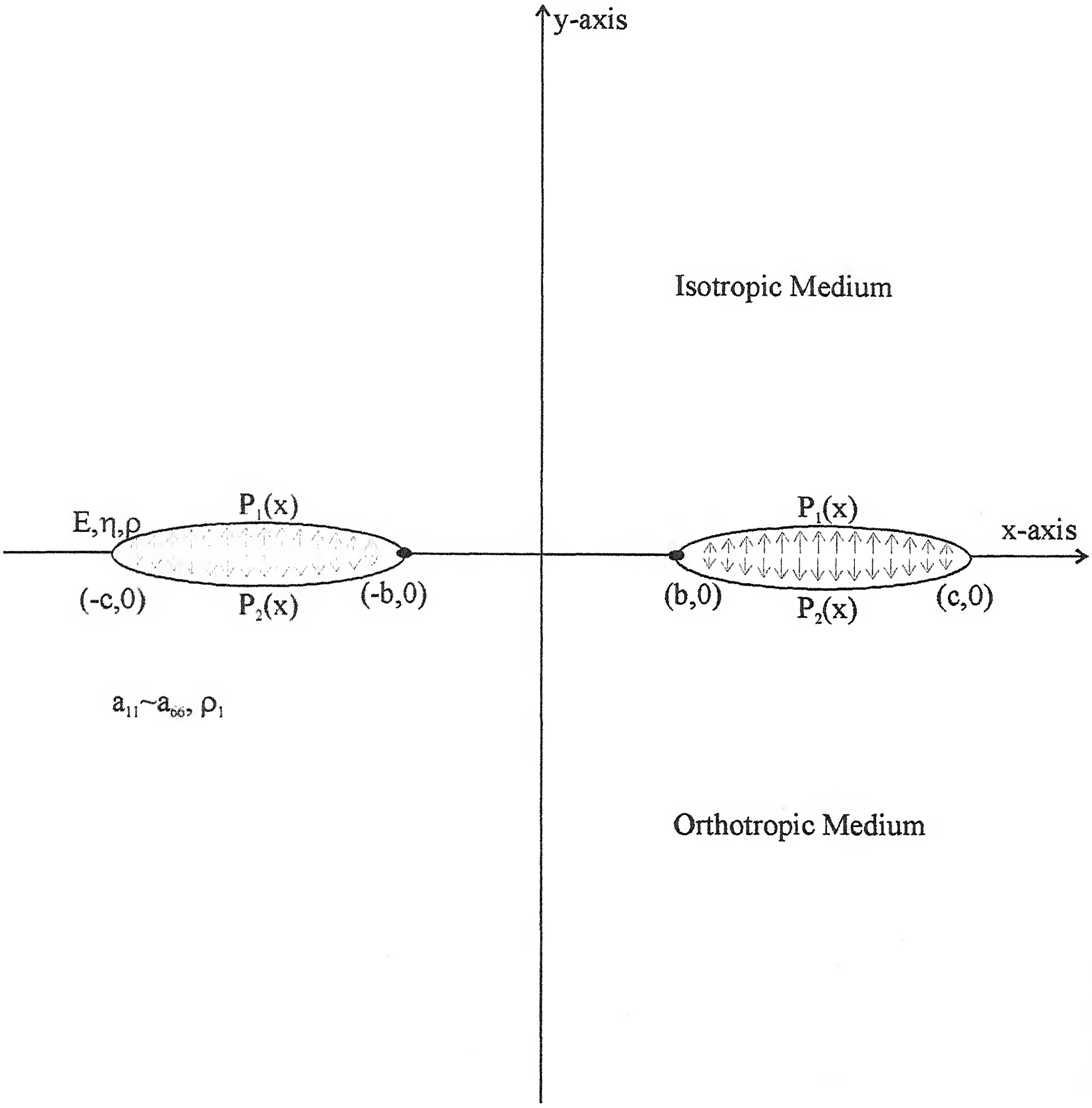

Fig. 7.2: Geometry with Boundary and Continuty Condition

$$\sigma_{yy}(x,0^{+}) = \sigma_{yy}(x,0^{-}) \qquad (7.1.6)$$

for $b<|x|<c$.

The following assumptions are due to the symmetry in crack geometry:

$$\sigma_{xy}(x,0^{+}) = \sigma_{xy}(x,0^{-}), \qquad b<x<c \qquad (7.1.7)$$

$$\sigma_{yy}(x,0^{+}) = \sigma_{yy}(x,0^{-}), \qquad b<x<c \qquad (7.1.8)$$

$$u_{y}(x,0^{+}) = u_{y}(x,0^{-}), \qquad b<x<c \qquad (7.1.9)$$

$$u_{x}(x,0^{+}) = u_{x}(x,0^{-}), \qquad b<x<c \qquad (7.1.10)$$

We concentrate analysis over Ist and IVth quadrant only. We checked through out that

$$u_{y}(x,0^{+})>0 \qquad (7.1.11)$$

$$u_{y}(x,0^{-})>0 \qquad (7.1.12)$$

which means that the cracks really open out.

The plan of the chapter is as follows: In section 7.2 the boundary value problem will be reduced to triple integral equations. In section 7.3, the solution of triple equations will be given in section 7.4, the approximate solution of coupled Fredholm integral equations of second kind will be given, in section 7.5, the physical quantities in the vicinity of crack tip will be given. The surface forces will be assumed as equal, constant and uniform.

7.2 REDUCTION TO TRIPLE INTEGRAL EQUATIONS

The component of stress and of displacement are assumed by means of the following equations (in the absence of body force).

For isotropic case (y>0)

$$\frac{\partial \sigma_{xx}}{\partial_x} + \frac{\partial \sigma_{yy}}{\partial_y} = 0 \tag{7.2.1}$$

$$\frac{\partial \sigma_{xy}}{\partial_x} + \frac{\partial \sigma_{yy}}{\partial_y} = 0 \tag{7.2.2}$$

$$u_y(x,y) = \frac{2(1+\eta)}{\pi E}\int_0^\infty \frac{\cos(\xi x)}{\xi^2}\left[\frac{(1-\eta)\partial^3}{\partial y^3}H(\xi,y) + (\eta-2)\xi^2\frac{\partial}{\partial y}H(\xi,y)\right]d\xi \tag{7.2.3}$$

$$u_x(x,y) = \frac{2(1+\eta)}{\pi E}\int_0^\infty \frac{\sin(\xi x)}{\xi}\left[(1-\eta)\frac{\partial^2 H}{\partial y^2} + \eta\xi^2 H(\xi,y)\right]d\xi \tag{7.2.4}$$

with

$$H(\xi,y) = [A(\xi) + yB(\xi)]e^{-\xi y} \tag{7.2.5}$$

where A and B are two arbitrary constants.

For orthotropic case (y<0)

$$u_y(x,y) = \int_0^\infty \frac{\cos(\xi x)}{\xi^2}\left[a_{11}G,yyy - \xi^2(a_{12}+a_{66})xG,y\right]d\xi \tag{7.2.6}$$

$$u_x(x,y) = \int_0^\infty \frac{\sin(\xi x)}{\xi}\left[a_{11}G,_{yy} - a_{12}\xi^2 G\right]d\xi \tag{7.2.7}$$

$$G(\xi, y) = \frac{1}{r_1 - r_2}\left[\{(r_1 - r_2)C(\xi) - D(\xi)\}e^{r_2 \xi y}\right] \tag{7.2.8}$$

where C and D are two arbitrary constants and r_1 an d r_2 are two roots of

$$r^4 + B_1 r^2 + B_2 = 0 \tag{7.2.9}$$

where

$$B_1 = \frac{2(a_{12} + a_{66})}{a_{11}}, \qquad B_2 = \frac{a_{22}}{a_{11}} \tag{7.2.10}$$

and $a_{11} \sim a_{66}$ etc. are elastic constants of orthotropic medium. The stress-strain relations are:

For isotropic case (y>0)

$$\left.\begin{aligned} 2\mu e_{xx} &= (1-\eta)\sigma_{xx} - \eta\sigma_{yy} \\ 2\mu e_{yy} &= (1-\eta)\sigma_{yy} - \eta\sigma_{xx} \\ 2\mu e_{xy} &= \sigma_{xy} \end{aligned}\right\} \tag{7.2.11}$$

where μ and η are modulus of rigidity and poisson's ratio of isotropic medium.

For orthotropic case (y<0)

$$\left.\begin{aligned}
\frac{\partial ux}{\partial x} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} \\
\frac{\partial uy}{\partial y} &= a_{12}\sigma_{xx} + a_{22}\sigma_{yy} \\
\frac{\partial ux}{\partial y} + \frac{\partial uy}{\partial x} &= a_{66}\sigma_{xy}
\end{aligned}\right\} \tag{7.2.12}$$

The boundary conditions (7.2.1)-7.2.2) and the continuity conditions (7.2.7)-(7.2.8), and then getting Fourier inversion of the functions given

$$\xi[r_1 C(\xi) - D(\xi)] + B(\xi) - \xi A(\xi) = 0 \tag{7.2.13}$$

$$A(\xi) - C(\xi) = P_1(\xi)/\xi^2 \tag{7.2.14}$$

with

$$\overline{P}_1(\xi) = \frac{1}{\pi}\int_b^c (p_1(x) - p_2(x))\cos(\xi x)dx \tag{7.2.15}$$

Now using the relations (7.2.3)-(7.2.8) and (7.1.9)-(7.1.10) we get the following integral relations:

$$\int_0^\infty \xi \sin(\xi x)[K_5 C(\xi) + K_6 D(\xi)]d\xi = P_4(x) \quad c \le x < \infty,\ 0 \le x \le b \tag{7.2.16}$$

$$\int_0^\infty \xi \cos(\xi x)[K_7 C(\xi) + K_8 D(\xi)]d\xi = P_5(x) \quad c \le x < \infty,\ 0 \le x \le b \tag{7.2.17}$$

with K as given by

$$\left.\begin{aligned}
K_5 &= K_0 - K_1 + 2K_0(1-\eta)(1-r_1) \\
K_6 &= K_2 - 2K_0(1-\eta) \\
K_7 &= K_0\{1 - 2(1-2\eta)(1-r_1)\} + K_3 \\
K_8 &= 2K_0(1-2\eta) + K_4 \\
K_0 &= \frac{2(1+\eta)}{\pi E}, K_1 = r_1^2 a_{11} - a_{12} \\
K_2 &= r_2 a_{11}, K_3 = r_1\left(a_{11} r_1^2 - a_{12} - a_{66}\right) \\
K_4 &= a_{11} r_2^2 - a_{12} - a_{66}
\end{aligned}\right\} \quad (7.2.18)$$

$$P_4 = \int_0^\infty \frac{\overline{P}_1(\xi)\sin(\xi x)d\xi}{\xi} \quad (7.2.19)$$

$$P_5 = \int_0^\infty \frac{\overline{P}_1(\xi)\cos(\xi x)d\xi}{\xi} \quad (7.2.20)$$

Similarly we now evaluate

$\sigma xy(x,0^+) = \sigma xy(x,0^-) = 0, \qquad b<x<c$

$\sigma yy(x,0^+) = \sigma yy(x,0^-) = -(P_1(x) + P_2(x)), \qquad b<x<c$

Thus above two relations along with (7.2.3)-(7.2.8) and the corresponding stress-strain relations lead us to

$$\int_0^\infty \xi^2 \sin(\xi x)[C(\xi)]d(\xi) = P_6(x) \qquad b < x < c \quad (7.2.21)$$

$$\int_0^\infty \xi^2 \cos(\xi x)[C(\xi)]d(\xi) = P_7(x) \qquad b < x < c \tag{7.2.22}$$

with

$$P_6(x) = \frac{1}{1+r_1}\int_0^\infty \overline{p}_1 \xi \sin(\xi x)dx \tag{7.2.23}$$

$$P_7(x) = -\frac{1}{2}\int_0^\infty \overline{p}_1 \xi \cos(\xi x) + \frac{1}{2}(p_1(x) + p_2(x)) \tag{7.2.24}$$

Thus the boundary value problem is reduced to the solution of triple integral equations (7.2.16)-(7.2.24).

7.3 SOLUTION OF TRIPLE INTEGRAL EQUAITONS

We assure that

$$\xi[K_5 C(\xi) + K_6 D(\xi)] = \phi(\xi) \tag{7.3.1}$$

$$\xi[K_7 C(\xi) + K_8 D(\xi)] = \psi(\xi) \tag{7.3.2}$$

where ϕ and ψ are two new functions

Then the integral equations (7.2.16)-(7.2.17); (7.2.21)-(7.2.22) become as new triple integral equations.

$$\int_0^\infty \phi(\xi)\sin(\xi x)d\xi = P_4(x), \qquad x \in I_1 \cup I_3 \tag{7.3.3}$$

$$\int_0^\infty \psi(\xi)\cos(\xi x)d\xi = P_5(x), \qquad x \in I_1 \cup I_3 \tag{7.3.4}$$

and

$$\int_0^\infty \xi[K_{12}\phi(\xi) + K_{13}\psi(\xi)]\sin(\xi x)d\xi = P_6(x), \qquad x \in I_2 \tag{7.3.5}$$

$$\int_0^\infty \xi[K_{10}\phi(\xi) - K_{11}\psi(\xi)]\cos(\xi x)d\xi = P_7(x), \qquad x \in I_2 \tag{7.3.6}$$

where K_{10} and K_{13} are given by (3.3.7) and

$$I_1 = [0,b],\ I_2 = [b,c],\ I_3 = [c,\infty] \tag{7.3.7}$$

Now to solve the system of equations (7.3.3)-(7.3.6) we make use of the method of Srivastava and Lowengrub [52]. We assume that

$$\pi\xi\phi(\xi) = 2\left[\int_b^c g(t)(1-\cos\xi t)dt - \left(\int_0^b + \int_c^\infty\right)P_4^1(t)(1-\cos\xi t)dt\right] \tag{7.3.8}$$

$$\pi\xi\phi(\xi) = 2\left[\int_b^c h(t)\sin\xi t\,dt - \left(\int_0^b + \int_c^\infty\right)P_5^1(t)\sin\xi t\,dt\right] \tag{7.3.9}$$

Then the equations (7.3.3)-(7.3.4) are satisfied identically if

$$\int_b^c g(t) = P_4(b) - P_4(c) \tag{7.3.10}$$

$$\int_b^c h(t) = P_5(b) - P_5(c) \tag{7.3.11}$$

Substitution of $\phi(\xi)$ and $\psi(\xi)$ from (7.3.8)-(7.3.9) into (7.3.5)-(7.3.6) and evaluating the integrals we get

$$K_{10}g(x) + \frac{2K_u}{\pi}\int_b^c \frac{th(t)dt}{t^2 - x^2} = -P_8(x), \qquad x \in I_2 \tag{7.3.12}$$

$$\frac{2}{\pi}K_{12}x\int_b^c \frac{g(t)dt}{t^2 - x^2} + K_{13}h(x) = P_8(x), \qquad x \in I_2 \tag{7.3.13}$$

where

$$P_8(x) = P_7(x) - \frac{4K_{11}}{\pi}\left(\int_0^b + \int_c^\infty\right)\frac{P_5(t)dt}{t^2 - x^2} \tag{7.3.14}$$

$$P_9(x) = P_6(x) + \frac{4K_{11}x}{\pi}\left(\int_0^b + \int_c^\infty\right)\frac{P_4(t)dt}{t^2 - x^2} \tag{7.3.15}$$

Now rewritting (7.3.12)-(7.3.13) into the following form as

$$\left.\begin{aligned} \int_b^c \frac{t\,h(t)dt}{t^2 - x^2} &= -\frac{\pi}{2K_{11}}P_{10}(x), \qquad x \in I_2 \\ \int_b^c \frac{x\,g(t)dt}{t^2 - x^2} &= \frac{\pi}{2K_{12}}P_{11}(x), \qquad x \in I_2 \end{aligned}\right\} \tag{7.3.16}$$

with

$$\left.\begin{aligned} P_{10}(x) &= P_8(x) + K_{10} g(x) \\ P_{11}(x) &= P_9(x) - K_{13} h(x) \end{aligned}\right\} \tag{7.3.17}$$

Inverting (7.3.16) by using Srivastava and Lowengrub method [52]

$$h(t) = \frac{1}{\pi K_{11}\delta(t)}\left[\int_b^c \frac{x\delta(x)P_{10}(x)dx}{x^2 - t^2} + D_1\right] \tag{7.3.18}$$

$$g(t) = \frac{1}{\pi K_{12}\delta(t)}\left[\int_b^c \frac{x\delta(x)P_{11}(x)dx}{x^2 - t^2} + D_2\right] \tag{7.3.19}$$

$$\delta(t) = \left\{\left(t^2 - b^2\right)\left(c^2 - t^2\right)\right\}^{1/2} \tag{7.3.20}$$

where D_1 and D_2 and two arbitrary constants which will obtained through (7.3.10)-(7.3.11) and (7.3.18)-(7.3.19). Substituting the values of $P_{10}(x)$ and $P_{11}(x)$ from (7.3.17) into (7.3.18)-(7.3.19) and evaluating the integrals.

$$h(t) = P_{12}(t) + \frac{K_{10}}{\pi K_{11}\delta(t)}\int_b^c \frac{x\delta(x)g(x)dx}{x^2 - t^2} \tag{7.3.21}$$

$$g(t) = P_{13}(t) - \frac{K_{13}t}{\pi K_{12}\delta(t)}\int_b^c \frac{\delta(x)h(x)dx}{x^2 - t^2} \tag{7.3.22}$$

$$P_{12}(t) = \left[D_1 + \int_b^c \frac{x\delta(x)P_8(x)dx}{x^2 - t^2}\right]\frac{1}{\pi K_{11}\delta(t)} \tag{7.3.23}$$

$$P_{13}(t) = \left[D_2 + \int_b^c \frac{\delta(x)P_9(x)dx}{x^2 - t^2}\right]\frac{1}{\pi K_{12}\delta(t)} \tag{7.3.24}$$

7.4 REDUCTION TO AND SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

We substitute for g(t) from (7.3.22), (7.3.24) into (7.3.20) and evaluate certain integrals, we get

$$h(t) = P_{14}(t) - \frac{K_{14}}{\delta(t)} \int_b^c \frac{\delta(\alpha) h(\alpha)}{\alpha^2 - t^2} H(t,\alpha) d\alpha \tag{7.4.1}$$

$$\begin{aligned} P_{14}(t) = P_{12}(t) + \frac{K_{15} D_2 [c - b + H_0(t)]}{2\delta(t)} \\ + \frac{K_{15}}{2\delta(t)} \int_b^c \left[\frac{\delta(\alpha) P_9(\alpha)}{\alpha^2 - t^2} \right] H^-(t,\alpha) d\alpha \end{aligned} \tag{7.4.2}$$

with

$$H_0(t) = \frac{1}{2} t \log \left| \frac{c-t}{c+t} \frac{b+t}{b-t} \right| \tag{7.4.3}$$

and

$$H(t,\alpha) = H_0(t) - H_0(\alpha) \tag{7.4.4}$$

$$\left. \begin{aligned} K_{14} &= \frac{K_{13} K_{10}}{2\pi^2 K_{11} K_{12}} \\ K_{15} &= \frac{K_{10}}{2\pi^2 K_{11} K_{12}} \end{aligned} \right\} \tag{7.4.5}$$

Similarly

$$g(t)=P_{15}(t)+\frac{t\,K_{16}}{\delta(t)}\int_b^c\frac{\delta(\alpha)h(\alpha)}{\alpha^2-t^2}\left[\frac{\phi_1(t)H_0^2(\alpha)}{\alpha^2}\right] \tag{7.4.6}$$

$$P_{15}(t)=\frac{1}{\alpha K_{12}\delta(t)}\left[D_{12}+\int_b^c\frac{\delta(x)P_9dx}{x^2-t^2}-K_{13}\int_b^c\frac{P_{14}(x)\delta(x)dx}{x^2-t^2}\right] \tag{7.4.7}$$

$$K_{16}=\frac{K_{13}K_{14}}{\pi K_{12}} \tag{7.4.8}$$

$$\phi_1(t)=\int_b^c\frac{H_0(x)dx}{x^2-t^2} \tag{7.4.9}$$

Thus knowing the solution of (7.4.1), we will substitute in (7.4.6) to get the solution of g(t).

Solution of Fredholm Integral Equation

We assume h(t) as

$$h(t)=\sum_{n=0}^{\infty}\delta^n h_n(t) \tag{7.4.10}$$

Substituting this value of h(t) in (7.4.1) and then comparing the coefficient of $\{\delta^n\}$ we get

$$\left.\begin{aligned}
H_0(t)&=P_{14}(t)\\
H_1(t)&=-K_{15}I_0(t)\\
H_2(t)&=-K_{15}I_1(t)+r_1K_{15}I_0(t)\\
H_3(t)&=-K_{15}I_2(t)+r_1K_{15}(1+K_{15})I_1(t)-r_1K_{15}I_0(t)
\end{aligned}\right] \tag{7.4.11}$$

where

$$I_n(t)=\int_b^c K_0(\alpha,t)\,I_{n-1}(\alpha)\,d\alpha \tag{7.4.12}$$

$$I_0(t)=\int_b^c K_0(\alpha,t)\,P_{14}\,d\alpha \tag{7.4.13}$$

Thus h(t) is written as

$$K_0(\alpha,t)=\frac{\delta(\alpha)H^-(t,\alpha)}{\delta(t)\left(\alpha^2-t^2\right)} \tag{7.4.14}$$

$$h(t)=P_{14}(t)-\delta K_{15}\Phi_2(t) \tag{7.4.15}$$

with

$$\Phi_2(t)=K_{20}I_0(t)+K_{21}I_1H+K_{22}I_2(t)$$

and

$$\left.\begin{aligned} K_{20}&=1-r_1\delta-r_1\delta^2\\ K_{21}&=\delta K_{15}-\delta^2 r_1\left(1+K_{15}\right)\\ K_{22}&=\delta^2 K_{15}\end{aligned}\right] \tag{7.4.16}$$

where I_0, I_1, I_2 etc. are given by (7.4.12)-(7.4.13).

Substituting this value of h(t) from (7.4.15)-(7.4.16) into (7.4.6) and evaluating certain integrals, we get

where

$$\Phi_3(t) = \phi_1(t)\Big[G_3^{(0)}(t) - \delta K_{15}K_{20}G_0^{(0)}(t) + \delta K_{15}K_{21}G_1^{(0)}(t)$$

$$+ \delta K_{15}K_{22}G_2^{(0)}(t)\Big] - G_3^{(1)}(t) - \delta K_{15}K_{21}G_1^{(1)}(t) \qquad (7.4.17)$$

$$+ \delta K_{15}K_{20}G_0^{(1)}(t) - \delta K_{15}K_{22}G_2^{(1)}(t)$$

and

$$\left.\begin{aligned} G_m^{(n)}(t) &= \int_b^c \frac{\delta(\alpha)}{\alpha^2 - t^2} I_m - \left\{\frac{H_0(\alpha)}{\alpha}\right\}^{2n} d\alpha \\ I_3(t) &= P_{14}(t) \end{aligned}\right] \qquad (7.4.18)$$

while

$\phi_1(t)$ is given by (7.4.9).

and the arbitrary constants D_1 and D_2 are evaluated through (7.4.15)-(7.4.18) and (7.3.10)-(7.3.11)

$$D_1 = \pi K_{11}[P_5(b) - P_5(c) - J_{33} + J_4]/J_{31}$$

$$- \frac{2\pi^2 K_{11}K_{15}J_{32}}{J_{32}}[P_4(b) - P_4(c) - K_{16}J_5] \qquad (7.4.19)$$

$$D_{12} = 2\pi[P_4(b) - P_4(c) - K_{16}J_5]K_{12} \qquad (7.4.20)$$

$J_{31} = \frac{1}{b}F(\pi/2, P_1)$ = Complete elliptic integral of first kind

$$K_1^2 = (c^2 - b^2)b^2$$

$$J_{32} = J_{31} + \int_b^c \frac{H_0(t)dt}{\delta(t)}$$

$$J_{33} = K_{15}\int_b^c \delta(\alpha)P_9(\alpha)d\alpha \int_b^c \frac{H^-(t,a)dt}{\delta(t)}$$

$$-\frac{1}{\pi K_{11}\left(c^2-b^2\right)}\int_b^c x\delta(x)P_8(x)\Pi\left[\frac{\pi}{2},K(x)\right]dx$$

Π is elliptic integral of third type

$$J_4 = \delta K_{15}\left[J_0K_{20} + K_{21}J_1 + K_{22}J_2\right]$$

$$J_5 = \delta K_{16}\left[M_3^{(0,1)} - \delta K_{15}K_{20}M_0^{(0,1)}\right]$$

$$+\delta K_{15}\left\langle \begin{matrix} K_{21}M_1^{(0,1)} + K_{22}M_2^{(0,1)} \\ K_{20}M_1^{(1,0)} - K_{21}M_1^{(1,0)} - M_2^{(1,0)} \end{matrix} \right\rangle - M_3^{(1,0)}$$

$$M_m^{(n,b)} = \int_b^c \frac{t\{\phi_1(t)\}^b}{\delta(t)} G_m^{(n)}(t)dt$$

$$J_m = \int_b^c I_m(t)dt, m = 1,2,3$$

7.5 PHYSICAL QUANTITITES

The quantities, which are important in fracture design criterions, are the components of stress in the neighbourhood of crack tips and the crack opening displacement. The components of stress are evaluated through the following relations.

Stress Components

Shear Stress

$$\sigma_{xy}\left(x,0^{+}\right)-\sigma_{xy}\left(x,0^{-}\right)=0, \qquad x\in I_1\cup I_3 \tag{7.5.1}$$

$$\sigma_{xy}\left(x,0^{+}\right)+\sigma_{xy}\left(x,0^{-}\right)=\left(1+r_1\right)\int_0^{\infty}\sin\xi x\left[\xi^2 C(\xi)-P_1(\xi)\right]d\xi \tag{7.5.2}$$

The above integral becomes as

$$\sigma_{xy}\left(x,0^{+}\right)+\sigma_{xy}\left(x,0^{-}\right)=\left(1+r_1\right)\int_0^{\infty}\xi x\left[K_{12}\phi(\xi)+K_{12}\psi(\xi)\right] \sin(\xi x)\,d\xi-P_6(x), \qquad x\in I_1\cup I_3 \tag{7.5.3}$$

Using relations (7.3.8)-(7.3.9) and then changing the order of integration and evaluating the integrals.

$$\sigma_{xy}\left(x,0^{+}\right)+\sigma_{xy}\left(x,0^{-}\right)=K_{12}\left(1+r_1\right)\left[\frac{2x}{\pi}\int_b^c\frac{g(t)dt}{t^2-x^2}-\frac{2x}{\pi}\left(\int_0^b+\int_c^{\infty}\right)\frac{P_4(t)dt}{t^2-x^2}\right]$$

$$-\left(1+r_1\right)K_{13}P_5(x)-P_6(x), \qquad x\in I_1\cup I_3 \tag{7.5.3}$$

Normal Stress

Similarly the normal component of stresses are given as

$$\sigma_{xy}\left(x,0^{+}\right)-\sigma_{xy}\left(x,0^{-}\right)=0, \qquad x\in I_1\cup I_3 \tag{7.5.5}$$

and

$$\sigma_{yy}\left(x,0^{+}\right)+\sigma_{yy}\left(x,0^{-}\right)=K_{10}P_{4}-\frac{2x}{\pi}\int_{b}^{c}\frac{th(t)dt}{t^{2}-x^{2}}$$

Now substituting for g(t) and h(t) from (7.4.15)-(7.4.18) into (7.5.4)-(7.5.6) and evaluating the integrals, we get

$$\sigma_{yy}\left(x,0^{+}\right)+\sigma_{xy}\left(x,0^{-}\right)=K_{12}(1+r_{1})\left[2x\langle P_{15}(x)+K_{16}\Phi_{3}(x)\rangle\frac{1}{\delta(x)}\right.$$

$$\left.\frac{-2x}{\pi}\left(\int_{0}^{b}+\int_{c}^{\infty}\right)\frac{P_{4}^{1}(t)dt}{t^{2}-x^{2}}\right]-(1+r_{1})K_{11}P_{5}^{1}(x)-P_{6}(x) \qquad x\in I_{1}$$

and

$$\sigma_{yy}\left(x,0^{+}\right)+\sigma_{xy}\left(x,0^{-}\right)=K_{12}(1+r_{1})\left[-2x\langle P_{15}(x)+K_{16}\Phi_{3}(x)\rangle\frac{1}{|\delta(x)|}\right.$$

$$\left.\frac{-2x}{\pi}\left(\int_{0}^{b}+\int_{c}^{\infty}\right)\frac{P_{4}^{1}(t)dt}{t^{2}-x^{2}}\right]-(1+r_{1})K_{13}P_{5}^{1}(x)-P_{6}(x) \qquad x\in I_{3}$$

and

$$\sigma_{yy}\left(x,0^{+}\right)+\sigma_{xy}\left(x,0^{-}\right)=\begin{cases}\dfrac{-2K_{11}}{|\delta(x)|}\{P_{14}(x)-\delta K_{15}\Phi_{3}(x)\}-f_{0}(x) & x\in I_{1}\\[2ex] \dfrac{2K_{11}}{|\delta(x)|}\{P_{14}(x)-\delta K_{15}\Phi_{2}(x)\}+f_{0}(x) & x\in I_{3}\end{cases}$$

$$|\delta(x)|=\left[\left|\left(b^{2}-x^{2}\right)\left(c^{2}-x^{2}\right)\right|\right]^{1/2}, \qquad x\in I_{1}$$

$$|\delta(x)|=\left[\left|\left(x^{2}-b^{2}\right)\left(x^{2}-c^{2}\right)\right|\right]^{1/2}, \qquad x\in I_{3}$$

$$F_0(x) = K_{10}P_4(x) + \frac{2}{\pi}\left(\int_0^b + \int_c^\infty\right) P_5^1(t)\frac{t\,dt}{t^2 - x^2}$$

$$+\frac{2}{\pi}K_{11}\left(\int_0^b + \int_c^\infty\right)\frac{P_5^1(t)\,t\,dt}{t^2 - x^2} - P_7(x) \qquad x \in I_1 \cup I_3 \tag{7.5.6}$$

Displacement Component

The displacement components are evaluated through relations (3.2.3);(7.2.2) - (7.2.3);(7.3.1) - (7.3.2);(7.3.8) - (7.3.9) and evaluating the integrals we get

iso $$u_y(x,0) = u_y(x,0^+) = \frac{2(1+\eta)}{\pi E}\frac{2(1+\eta)P_5(x)}{\pi E}$$

$$+\frac{K_{10}^+}{\pi}\left[\left\langle \int_b^c g(t) - \left(\int_0^b + \int_c^\infty\right) P_4^1(t)\right\rangle \log\left|x^2 - t^2\right| dt\right]$$

$$+K_{11}^+\left[\int_b^c h(t)dt - P_4(c)\right] \qquad x \in I_2$$

with

$$\left.\begin{aligned} K_{10}^+ &= K_0\left[\frac{r_1K_8 - K_7 - r_1K_6}{K_9}\right] \\ K_{11}^+ &= K_0\left[\frac{(1-2\eta)K_5 - r_1K_6}{K_9}\right] \end{aligned}\right\} \tag{7.5.8}$$

Further

orth. $$u_y(x,0) = u_y(x,0^+) = \int_0^\infty \cos(\xi x)\left[\xi\langle K_3 C(\xi) + K_4 D(\xi)\rangle\right] d\xi$$

$$= \int_0^\infty \cos(\xi x)\left[K_3 \frac{K_8\phi - K_6\psi}{K_9} - \frac{K_4}{K_9}(K_7\phi - K_5\psi)\right]$$

$$= \int_0^\infty \cos(\xi x)\left[\phi\langle K_9(K_8K_3 - K_4K_7)\rangle + \frac{K_4K_5 - K_3K_6}{K_9}\psi\right] d\xi$$

Thus

$$u_y(x,0^-) = \int_0^\infty \cos(\xi x)\left[\phi K_{13}^+ + K_{14}^+\psi\right] d\xi \tag{7.5.9}$$

$$\left.\begin{aligned} K_{10}^+ &= \frac{K_3K_8 - K_4K_7}{K_9} \\ K_{14}^+ &= \frac{K_4K_5 - K_3K_6}{K_9} \end{aligned}\right\} \tag{7.5.10}$$

$$\begin{aligned} u_y(x,0^-) = P_5(x) + \frac{K_{13}^*}{\pi}&\left[\int_b^c g(t)\log\left|x^2 - t^2\right| dt\right. \\ &\left. - \left(\int_0^b + \int_c^\infty\right) P_4^1(t)\log\left|x^2 - t^2\right| dt\right] \\ &+ K_{11}^+\left[\int_x^c h(t)dt + P_5(c)\right], \qquad x \in I_2 \end{aligned} \tag{7.5.11}$$

Similarly

$$u_x\left(x,0^+\right) = P_4(x)K_0 + K_{16}^+\left[\int_x^c g(t)dt + P_4(c)\right]$$

$$+\frac{K_{17}^+}{\pi}\left[\int_b^c h(t) - \left(\int_b^c + \int_c^\infty\right)P_5(t)\right)\log\left|\frac{x-t}{x+t}\right|dt\right] \qquad x \in I_2 \tag{7.5.12}$$

with

$$\left.\begin{aligned} K_{16}^+ &= K_0\left[\frac{K_{15}^+K_8 + 2(1-\eta)K_2}{K_9}\right] \\ K_{17}^+ &= K_0\left[\frac{K_{15}^+K_6 + 2(1-\eta)K_5}{K_9}\right] \\ K_{15}^+ &= 1 + 2(1-\eta)(1-r_1) \end{aligned}\right| \tag{7.5.13}$$

$$u_x\left(x,0^-\right) = \int_0^\infty \xi\cos(\xi x)\left[C(\xi) - \frac{a_{11}r_2D}{K_2}\right]d\xi + K_0P_4(x)$$

$$= \int_0^\infty \sin(\xi x)\left[K_1\left\langle\frac{K_8\phi - K_6\psi}{K_9}\right\rangle + K_2\frac{K_7\phi - K_5\psi}{K_9}\right]d\xi$$

$$= \int_0^\infty \sin(\xi x)\left[K_{18}^+\phi(\xi) - K_{19}^+\psi(\xi)\right]d\xi + K_0P_4(x), \qquad x \in I_2$$

$$\begin{aligned} K_{18}^+ &= \frac{K_1K_8 - K_2xK_7}{K_9} \\ K_{19}^+ &= \frac{K_1K_6 - K_2xK_5}{K_9} \end{aligned} \tag{7.5.14}$$

Thus

$$u_x\left(x,0^-\right)=K_0P_4(x)+K_{16}^{+}\left[\int_x^c g(t)dt+P_4(c)\right]$$
$$+\frac{K_{19}^{+}}{\pi}\left[\int_b^c h(t)-\left(\int_0^b+\int_c^\infty\right)P_5(t)\rangle\log\left|\frac{x-t}{x+t}\right|dt\right] \qquad x\in I_2 \tag{7.5.15}$$

7.6 SPECIAL CASE

The given forces at crack faces are assumed to be constant, uniform and equal, see figure 7.3

$$P_1(x)=P_2(x)=P_0 \tag{7.6.1}$$

then

$$P_1(\xi)=0 \tag{7.6.2}$$

and

$$P_4(x)=P_5(x)=P_6(x)=0 \tag{7.6.3}$$

$$P_7(x)=P_0 \tag{7.6.4}$$

then (7.3.10)-(7.3.11) reduce

$$\left.\begin{aligned}&\int_b^c g(t)dt=0,\\&\int_b^c h(t)dt=0,\end{aligned}\right\} \tag{7.6.5}$$

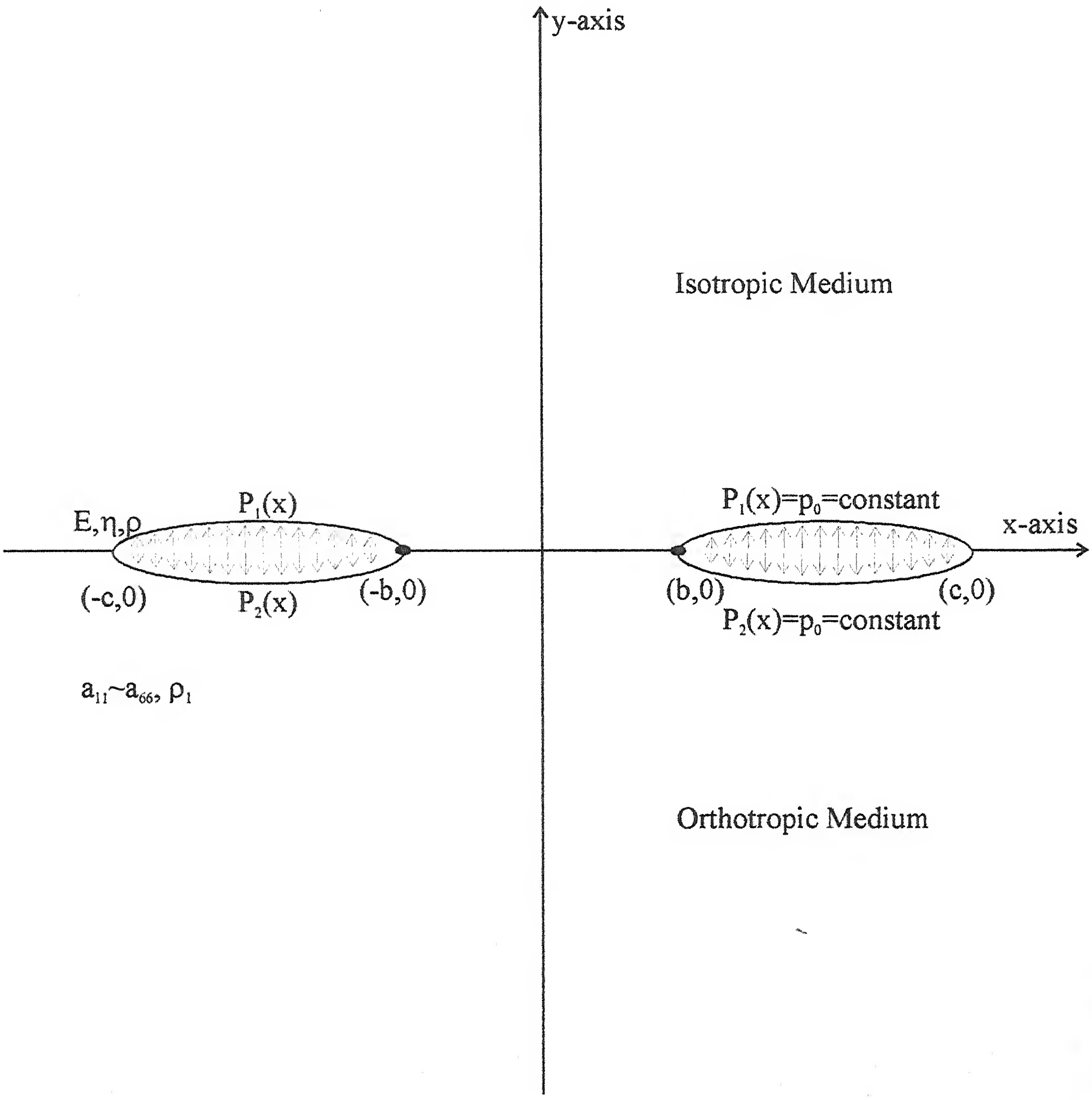

Fig. 7.3: Geometry of Problem with Special Force

$$P_8(x) = P_0 \tag{7.6.6}$$

$$P_9(x) = P_0 \tag{7.6.7}$$

$$P_{10}(x) = K_{10}g(x) + P_0 \tag{7.6.8}$$

$$P_{11}(x) = -K_{13}h(t) \tag{7.6.9}$$

$$P_{12}(t) = \left[D_1 + p_0\left\{2t^2 - c^2 - b^2\right\}\frac{\pi}{2}\right]\frac{1}{\pi K_{11}\delta(t)} \tag{7.6.10}$$

$$P_{13}(t) = \frac{t\,D_2}{\pi K_{11}\delta(t)} \tag{7.6.11}$$

$$P_{14}(t) = P_{12}(t) + \frac{K_{11}\,D_2}{\delta(t)}\{c - b + H_0(t)\} \tag{7.6.12}$$

$$P_{15}(t) = \frac{t}{\pi K_{12}\delta(t)}\left[D_2 - \frac{K_{13}}{\pi K_{11}}\left\langle\left\{D_1 + \frac{\pi p_0}{2}\left(c^2 - b^2\right)\right\}\right.\right.$$
$$\left.\frac{H_0(t)}{2t^2} + \pi p_0\langle c - b + H_0(t)\rangle\right\rangle \tag{7.6.13}$$

where (7.6.3)-(7.6.7) are substituted in (7.4.19)-4.4.21), we get

$$D_1 = \frac{\pi K_{11}}{J_{11}}[J_4 - J_{33}] + 2\pi^2 K_{11}K_{12}\frac{K_{15}K_{16}J_{32}J_5}{J_{31}}$$

$$J_{33} = \frac{p_0}{\pi K_{11}}\left[\pi - 2b^2 J_{31}\right] \tag{7.6.14}$$

The constants J_4, J_5, J_{32}, J_{31} can be evaluated numerically, through these integrals which are to be handled carefully for numerical integrations.

$$D_2 = -2\pi K_{12} K_{16} J_5 \tag{7.6.15}$$

The constant J_{32} has logarithmic oscillations, hence the evaluation has to be done very carefully. The functions h(t) and g(t) given by (7.4.15), (7.4.16) with P_{14}, P_{15} given by (7.6.12), (7.6.13) and constants D_1 and D_2 the there in by (7.6.14)-(7.6.15) and remaining constants and functions as defined previously.

Thus the physical quantities can be evaluated by numerical integration.

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